Converges in Neutrosophic Metric Spaces

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ABSTRACT

Following Kirişci and Şimşek are definitions of a neutrosophic metric space. We have established related results and illustrated them with examples; also discuss membership between neutrosophic metric space and classical metric space. Additionally, study the concepts of convergence and Cauchy sequences. Finally, normally and separably within a neutrosophic metric space.

1. Introduction

The notion of a neutrosophic metric space (NMS) was introduced by Kirişci and Şimşek [5] in 2020 as a novel metric space that naturally expands fuzzy metric space (FMS) [7] and intuitionistic fuzzy metric space (IFMS) [6]. This was achieved by employing the neutrosophic set, an idea originally proposed by [3], F. Smarandache, as an extension of fuzzy sets introduced in 1965 by Zadeh [1]. In 1986, Atanassov [2] invented the intuitionistic fuzzy set. Zadeh introduced fuzzy set theory, which evolves from classical sets and is predicated on the notion that the degree of membership or truth (T) for each element within the set takes a value in the interval [0,1]. This framework is one of the most potent instruments in contemporary mathematics, facilitating the modeling and enhancement of various scientific disciplines reliant upon it. Atanassov was the pioneer in defining and introducing the degree of membership (T) and non- membership or untruth (F), referred to as the intuitionistic fuzzy set. In the theory of intuitionistic fuzzy sets (IFS), the degree of non- membership is not analogous to the complement of the degree membership due to the absence of information when these degrees of are unequal. Actually, intuitionistic fuzziness came up accidentally in the field of mathematics. Combining the degrees of membership and non- membership yields one, which Atanassov used to expand the conventional fuzzy set. While Smarandache's neutrosophic theory realizes the neutrosophic set on three components (truth, indeterminacy, and falsehood), T, I and F are typically subsets of the interval [0,1]. Recently, there have been many research works on neutrosophic metric space; for example, Ghosh et al. introduced a definition of neutrosophic fuzzy metric space in 2024 and studied some properties and concepts, including open ball, Hausdorrf, and boundedness. Abdullah [9], in 2025, derived the definition of neutrosophic partial metric spaces; he studied the properties of convergence and fixed points. Below is an outline of the paper's structure. Characteristics and core ideas behind neutrosophic sets and neutrosophic metric spaces with convergent sequences, Cauchy sequences, open and closed balls, and open sets are provided in section 2. In section 3, introduce examples of NMS, investigate their properties, and demonstrate that every NMS is induced by a metric space. Also study convergence and provide normality and separability.

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2. Preliminaries

We begin with certain definitions, examples, lemmas, and properties that are essential for understanding the main results before moving on to the next section.

Definition 2.1 [1]. Let ϒ a universal set, a Fuzzy set (FS) $\bar{E}_{FS}$ is characterized by the set

$$\bar{E}_{FS} = \{\langle g, Ɣ_{FS}(g)\rangle : 0 \leq Ɣ_{FS}(g) \leq 1,\ g \in ϒ\}, \text{ where}$$

$Ɣ_{FS}: ϒ \longrightarrow [0,1]$ is the membership function of $\bar{E}_{FS}$ and $Ɣ_{FS}(g)$ is signifies the membership grade of g in $\bar{E}_{FS}$.

Definition 2.2 [2]. Let $\bar{E}_{IFS}$ be an Intuitionistic fuzzy set (IFS) over the universal set ϒ, defined as an object of the form

$\bar{E}_{IFS} = \{\langle g, Ɣ_{IFS}(g), Ƒ_{IFS}(g)\rangle :\ g \in ϒ\}$, where $Ɣ_{IFS}, Ƒ_{IFS}: ϒ \longrightarrow [0,1]$ are, respectively, the membership and non-membership functions of the elements $g \in ϒ$, and for

$$0 \leq Ɣ_{IFS}(g) + Ƒ_{IFS}(g) \leq 1, \forall\ g \in ϒ.$$

The quantity

$$\pi_{IFS}(g) \coloneqq 1 - Ɣ_{IFS}(g) - Ƒ_{IFS}(g),$$

represents the hesitation degree at g.

Intuitively, an ordinary FS corresponds to an IFS with non-membership degree $Ƒ_{IFS}(g) = 1 - Ɣ_{IFS}(g)$ for all $g \in ϒ$, that is when $\pi_{IFS}(g) = 0, \forall\ g$.

Definition 2.3 [3]. A Neutrosophic set (NS) $\bar{E}$ relative to a universal set ϒ is defined as

$$\bar{E} = \{\langle g, (C_N(g), B_N(g), S_N(g))\rangle :\ g \in ϒ\}$$

Where the functions

$$C_N, B_N, S_N : ϒ \longrightarrow [0,1].$$

Here, $C_N(g), B_N(g)$ and $S_N(g)$ denote, respectively, the truth, indeterminacy, and untruth/falsity membership grades of g in set $\bar{E}$. These values satisfy the following condition:

$$0 \leq C_N(g) + B_N(g) + S_N(g) \leq 3.$$

Definition 2.4 [4]. A binary operation $\boxtimes: [0,1]^2 \longrightarrow [0,1]$ is a continuous t-norm (CTN) if $\boxtimes$ satisfies the following properties for all $\tilde{α}, \tilde{η}, \tilde{υ}, \tilde{ω} \in [0,1]$:

1. $\tilde{α} \boxtimes 1 = \tilde{α}$;
2. $\tilde{α} \boxtimes \tilde{η} \leq \tilde{υ} \boxtimes \tilde{ω}$ *whenever* $\tilde{α} \leq \tilde{υ}$ and $\tilde{η} \leq \tilde{ω}$;
3. $\boxtimes$ is continuous;
4. $\boxtimes$ is commutative and associative. As an illustration,

$\tilde{α} \boxtimes \tilde{η} = \tilde{α}\tilde{η}$, $\tilde{α} \boxtimes \tilde{η} = min\{\tilde{α}, \tilde{η}\}$, $\tilde{α} \boxtimes \tilde{η} = max\{\tilde{α} + \tilde{η} - 1, 0\}$.

Definition 2.5 [4]. A binary operation $\boxdot: [0,1]^2 \longrightarrow [0,1]$ is a continuous t-conorm (CTCN) if $\boxdot$ satisfies the following properties for all $\tilde{α}, \tilde{η}, \tilde{υ}, \tilde{ω} \in [0,1]$:

1. $\tilde{α} \boxdot 0 = \tilde{α}$;
2. $\tilde{α} \boxdot \tilde{η} \leq \tilde{υ} \boxdot \tilde{ω}$ *whenever* $\tilde{α} \leq \tilde{υ}$ and $\tilde{η} \leq \tilde{ω}$;
3. $\boxdot$ is continuous;
4. $\boxdot$ is commutative and associative. As an illustration,

$\tilde{α} \boxdot \tilde{η} = \tilde{α} + \tilde{η} - \tilde{α}\tilde{η}$, $\tilde{α} \boxdot \tilde{η} = max\{\tilde{α}, \tilde{η}\}$, $\tilde{α} \boxdot \tilde{η} = min\{\tilde{α} + \tilde{η}, 1\}$.

Remark 2.6 [6]. (1) For any $ȁ, ȕ \in [0,1]$, if $ȁ > ȕ$ we can find $ȑ, ȅ \in [0,1]$ such that $ȁ \boxtimes ȑ \geq ȕ$ and $ȁ \geq ȕ \boxdot ȅ$.

(2) if ȁ ∈ [0,1], there are ȕ, ȑ ∈ [0,1] such that ȕ ⊠ ȕ ≥ ȁ and ȁ ≥ ȑ ⊡ ȑ.

Definition 2.7 [5]. Let ϒ be a non-empty set and Ē be an NS such that $\bar{E}: \Upsilon \times \Upsilon \times \mathbb{R}^+ \longrightarrow I$. Let ⊠ be (CTN) and ⊡ be (CTCN). Then the 4-tuple (ϒ, Ē ,⊠ ,⊡) is called *Neutrosophic Metric Space* (NMS) on ϒ, subject to the following axioms. For the ᾅ, ῄ, ῴ ∈ ϒ and $\zeta, \eta \in \mathbb{R}^+$

1. $0 \leq C_N(ᾅ, ῄ, \zeta),\ B_N(ᾅ, ῄ, \zeta),\ S_N(ᾅ, ῄ, \zeta) \leq 1$;
2. $C_N(ᾅ, ῄ, \zeta) + B_N(ᾅ, ῄ, \zeta) + S_N(ᾅ, ῄ, \zeta) \leq 3$;
3. $C_N(ᾅ, ῄ, \zeta) = 1\ iff\ ᾅ = ῄ$;
4. $C_N(ᾅ, ῄ, \zeta) = C_N(ῄ, ᾅ, \zeta)$;
5. $C_N(ᾅ, ῄ, \zeta + \eta) \geq C_N(ᾅ, ῴ, \zeta) ⊠ C_N(ῴ, ῄ, \eta)$;
6. $C_N(ᾅ, ῄ, .) : \mathbb{R}^+ \longrightarrow \mathbb{I}\ is\ continuous$;
7. $\lim_{i \to \infty} C_N(ᾅ, ῄ, \zeta) = 1$;
8. $B_N(ᾅ, ῄ, \zeta) = 0\ iff\ ᾅ = ῄ$;
9. $B_N(ᾅ, ῄ, \zeta) = B_N(ῄ, ᾅ, \zeta)$;
10. $B_N(ᾅ, ῄ, \zeta + \eta) \leq B_N(ᾅ, ῴ, \zeta) ⊡ B_N(ῴ, ῄ, \eta)$;
11. $B_N(ᾅ, ῄ, .) : \mathbb{R}^+ \longrightarrow \mathbb{I}\ is\ continuous$;
12. $\lim_{i \to \infty} B_N(ᾅ, ῄ, \zeta) = 0$;
13. $S_N(ᾅ, ῄ, \zeta) = 0\ iff\ ᾅ = ῄ$;
14. $S_N(ᾅ, ῄ, \zeta) = S_N(ῄ, ᾅ, \zeta)$;
15. $S_N(ᾅ, ῄ, \zeta + \eta) \leq S_N(ᾅ, ῴ, \zeta) ⊡ S_N(ῴ, ῄ, \eta)$;
16. $S_N(ᾅ, ῄ, .) : \mathbb{R}^+ \longrightarrow \mathbb{I}\ is\ continuous$;
17. $\lim_{i \to \infty} S_N(ᾅ, ῄ, \zeta) = 0$;
18. $If\ \zeta \leq 0,\ then\ C_N(ᾅ, ῄ, \zeta) = 0,\ B_N(ᾅ, ῄ, \zeta) = 1,\ S_N(ᾅ, ῄ, \zeta) = 1$.

Then $\bar{E} = (C, B, S)$ is called N*eutrosophic* Me*tric* (NM)on ϒ.

For a given ζ, $C_N(ᾅ, ῄ, \zeta)$, $B_N(ᾅ, ῄ, \zeta)$, and $S_N(ᾅ, ῄ, \zeta)$ measure the degree of nearness, neutrality, and non- nearness between ᾅ and ῄ.

3. Principal Results

Example 3.1. Let $\Upsilon = \mathbb{N}$. Define $\tilde{\alpha} ⊠ \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$, $\tilde{\alpha} ⊡ \tilde{\eta} = \tilde{\alpha} + \tilde{\eta} - \tilde{\alpha}\tilde{\eta}$ for all $\tilde{\alpha}, \tilde{\eta} \in [0,1]$ and define $C_N, B_N, S_N : \Upsilon^2 \times \mathbb{R}^+ \longrightarrow [0,1]$ by

$$C_N(ṣ, ṛ, \zeta) = \begin{cases} \dfrac{ṣ}{ṛ}, & ṣ \leq ṛ \\ \dfrac{ṛ}{ṣ}, & ṛ \leq ṣ \end{cases}, B_N(ṣ, ṛ, \zeta) = \begin{cases} \dfrac{ṛ - ṣ}{cṛ}, & ṣ \leq ṛ \\ \dfrac{ṣ - ṛ}{cṣ}, & ṛ \leq ṣ \end{cases}, S_N(ṣ, ṛ, \zeta) = \begin{cases} \dfrac{ṛ - ṣ}{ṛ}, & ṣ \leq ṛ \\ \dfrac{ṣ - ṛ}{ṣ}, & ṛ \leq ṣ \end{cases}$$

for all ṣ, ṛ ∈ ϒ, and $\zeta, c > 0$. Then (ϒ, Ē,⊠,⊡) a neutrosophic metric space.

Proof.

1) $\forall\ \zeta > 0,\ and\ ṣ, ṛ \in \Upsilon, then\ \dfrac{ṣ}{ṛ}, \dfrac{ṛ}{ṣ} > 0 \implies C_N(ṣ, ṛ, \zeta) > 0.$

2) $\forall\ \zeta > 0,\ and\ ṣ, ṛ \in \Upsilon, then\ \dfrac{ṣ}{ṛ} = \dfrac{ṛ}{ṣ} = 1\ iff\ ṣ = ṛ \implies C_N(ṣ, ṛ, \zeta) = 1\ iff\ ṣ = ṛ.$

3) $\forall\, \zeta > 0,\ \ and\ ṣ, ṛ \in \Upsilon\ \ then\ C_N(ṣ, ṛ, \zeta) = \begin{cases} \frac{ṣ}{ṛ}\ ,\ \ ṣ \le ṛ \\ \\ \frac{ṛ}{ṣ}\ ,\ \ ṛ \le ṣ \end{cases} = \begin{cases} \frac{ṛ}{ṣ}\ ,\ \ ṛ \le ṣ \\ \\ \frac{ṣ}{ṛ}\ ,\ \ ṣ \le ṛ \end{cases} = C_N(ṛ\,, ṣ, \zeta).$

4) $Let\ ṣ, ṛ, ṭ \in \Upsilon\ , and\ \zeta, \eta > 0,$ to prove $C_N(ṣ, ṛ, \zeta + \eta) \ge C_N(ṣ, ṭ, \zeta) \boxtimes C_N(ṭ, ṛ, \eta).$

There are four cases:

$i)\ If\ \ ṣ = ṛ = ṭ,\ \ then\ C_N(ṣ, ṛ, \zeta + \eta) = C_N(ṣ, ṭ, \zeta) = C_N(ṭ, ṛ, \eta) = 1\ implies$

$$C_N(ṣ, ṛ, \zeta + \eta) \ge C_N(ṣ, ṭ, \zeta) \boxtimes C_N(ṭ, ṛ, \eta).$$

$ii)\ If\ \ ṣ = ṛ \ne ṭ,$ for simplicity, assume that, $ṣ = ṛ$ and $ṛ < ṭ$, thus $ṣ < ṭ$, then

$$C_N(ṣ, ṛ, \zeta) = 1\ ,\ \ C_N(ṛ, ṭ, \eta) = \frac{ṛ}{ṭ}\ ,\quad C_N(ṣ, ṭ, \zeta + \eta) = \frac{ṣ}{ṭ}$$

Now,

$$1 \times \frac{ṛ}{ṭ} = \frac{ṛ}{ṭ}\ \Longrightarrow\ 1 \times \frac{ṛ}{ṭ} = \frac{ṣ}{ṭ}\ \ implies\ C_N(ṣ, ṛ, \zeta) \boxtimes C_N(ṛ, ṭ, \eta) \le C_N(ṣ, ṭ, \zeta + \eta).$$

$iii)\ If\ \ ṣ \ne ṛ = ṭ,$ for simplicity, assume that, $ṣ < ṛ$ and $ṛ = ṭ$, thus $ṣ < ṭ$, then

$$C_N(ṣ, ṛ, \zeta) = \frac{ṣ}{ṛ}\ ,\quad C_N(ṛ, ṭ, \eta) = 1\ ,\quad C_N(ṣ, ṭ, \zeta + \eta) = \frac{ṣ}{ṭ}$$

Now,

$$\frac{ṣ}{ṛ} \times 1 = \frac{ṣ}{ṛ}\ \Longrightarrow\ \frac{ṣ}{ṛ} \times 1 = \frac{ṣ}{ṭ}\ \ implies\ C_N(ṣ, ṛ, \zeta) \boxtimes C_N(ṛ, ṭ, \eta) \le C_N(ṣ, ṭ, \zeta + \eta).$$

$iv)\ If\ \ ṣ \ne ṛ \ne ṭ,$ for simplicity, assume that, $ṣ < ṛ\ < ṭ$, thus

$$C_N(ṣ, ṛ, \zeta + \eta) = \frac{ṣ}{ṛ}\ ,\ \ C_N(ṛ, ṭ, \eta) = \frac{ṛ}{ṭ}\ ,\quad C_N(ṣ, ṭ, \zeta) = \frac{ṣ}{ṭ}$$

Now,

$$ṛ < ṭ\quad \Longrightarrow\quad ṛ^2 < ṭ^2\quad \Longrightarrow\quad \frac{1}{ṭ^2} < \frac{1}{ṛ^2}\quad \Longrightarrow\quad \frac{ṣṛ}{ṭ^2} < \frac{ṣṛ}{ṛ^2}\quad \Longrightarrow\quad \frac{ṣ}{ṭ} \times \frac{ṛ}{ṭ} < \frac{ṣ}{ṛ}\ \ implies$$

$$C_N(ṣ, ṭ, \zeta) \boxtimes C_N(ṛ, ṭ, \eta) \le C_N(ṣ, ṛ, \zeta + \eta)\ \Longrightarrow\ C_N(ṣ, ṭ, \zeta) \boxtimes C_N(ṭ, ṛ, \eta) \le C_N(ṣ, ṛ, \zeta + \eta).$$

5) $Since\ C_N(ṣ, ṭ, .): \mathbb{R}^+ \longrightarrow [0,1]\ is\ continuous, becuse\ C_N(ṣ, ṭ, \zeta)\ is\ idependent\ of\ \zeta.$

Now, respect to $B_N(ṣ, ṭ, \zeta)$ we shall only verify the axiom (10). Others are straightforward.

There are many cases:

$i)\ If\ \ ṣ = ṛ = ṭ,\ \ then\ B_N(ṣ, ṛ, \zeta + \eta) = B_N(ṣ, ṭ, \zeta) = B_N(ṭ, ṛ, \eta) = 0.$

Now,

$$0 = 0 + 0 - 0 \times 0 = 0 \boxdot 0\quad implies\quad B_N(ṣ, ṛ, \zeta + \eta) \le B_N(ṣ, ṭ, \zeta) \boxdot B_N(ṭ, ṛ, \eta).$$

$ii)\ If\ \ ṣ = ṛ \ne ṭ,$ for simplicity, assume that $ṣ = ṛ$ and $ṛ < ṭ$, thus $ṣ < ṭ$, then

$$B_N(ṣ, ṛ, \zeta) = 0\ ,\ \ B_N(ṛ, ṭ, \eta) = \frac{ṭ - ṛ}{cṭ}\ ,\quad B_N(ṣ, ṭ, \zeta + \eta) = \frac{ṭ - ṣ}{cṭ}.$$

Now,

$$\frac{ṭ - ṣ}{cṭ} = 0 + \frac{ṭ - ṣ}{cṭ} - (0) \times \left(\frac{ṭ - ṣ}{cṭ}\right)\quad \Longrightarrow\quad \frac{ṭ - ṣ}{cṭ} = 0 + \frac{ṭ - ṛ}{cṭ} - (0) \times \left(\frac{ṭ - ṛ}{cṭ}\right)$$

$implies\ B_N(ṣ, ṭ, \zeta + \eta) \le B_N(ṣ, ṛ, \zeta) \boxdot B_N(ṛ, ṭ, \eta).$

$iii)\ If\ \ ṣ \ne ṛ = ṭ,$ for simplicity, asume that $ṣ < ṛ$ and $ṛ = ṭ$, thus $ṣ < ṭ$, then

$$B_N(ṣ,ṛ,\zeta) = \frac{ṛ - ṣ}{cṛ}, \quad B_N(ṛ,ṭ,\eta) = 0, \quad B_N(ṣ,ṭ,\zeta+\eta) = \frac{ṭ - ṣ}{cṭ}.$$

Now,

$$\frac{ṭ - ṣ}{cṭ} = 0 + \frac{ṭ - ṣ}{cṭ} - (0) \times \left(\frac{ṭ - ṣ}{cṭ}\right) \quad \Longrightarrow \quad \frac{ṭ - ṣ}{cṭ} = 0 + \frac{ṛ - ṣ}{cṛ} - (0) \times \left(\frac{ṛ - ṣ}{cṛ}\right)$$

$implies\ B_N(ṣ,ṭ,\zeta+\eta) \le B_N(ṣ,ṛ,\zeta) \boxdot B_N(ṛ,ṭ,\eta)$.

$iv)$ If $ṣ \ne ṛ \ne ṭ$; for simplicity, assume that, $ṣ < ṛ < ṭ$, thus

$$B_N(ṣ,ṛ,\zeta+\eta) = \frac{ṛ - ṣ}{cṛ}, \quad B_N(ṛ,ṭ,\eta) = \frac{ṭ - ṛ}{cṭ}, \quad B_N(ṣ,ṭ,\zeta) = \frac{ṭ - ṣ}{cṭ}$$

Now,

$B_N(ṣ,ṭ,\zeta) \boxdot B_N(ṭ,ṛ,\eta) - B_N(ṣ,ṛ,\zeta+\eta) \quad \Longrightarrow$

$$= \left(\frac{ṭ - ṣ}{cṭ}\right) + \left(\frac{ṭ - ṛ}{cṭ}\right) - \left(\frac{ṭ - ṣ}{cṭ}\right) \times \left(\frac{ṭ - ṛ}{cṭ}\right) - \left(\frac{ṛ - ṣ}{cṛ}\right) \ge 0 \quad \Longrightarrow$$

$B_N(ṣ,ṛ,\zeta+\eta) \le B_N(ṣ,ṭ,\zeta) \boxdot B_N(ṭ,ṛ,\eta)$.

Similarly to $S_N(ṣ,ṛ,\zeta)$ Therefore $(\Upsilon, \bar{\text{E}}, \boxtimes, \boxdot)$ is an NMS.

$Remark$ 3.2. The last example is not an NMS with (CTN) defined by $\tilde{\alpha} \boxtimes \tilde{\eta} = \mathfrak{min}\{\tilde{\alpha}, \tilde{\eta}\}$ & (CTCN) defined by $\tilde{\alpha} \boxdot \tilde{\eta} = \mathfrak{max}\{\tilde{\alpha}, \tilde{\eta}\}$. To show that, for B_N, taking $ṣ = 1,\ ṛ = 2,\ ṭ = 3$,

$$\forall\ \zeta,\eta > 0, then \quad B_N(ṣ,ṛ,\zeta) = \frac{1}{2}, \quad B_N(ṛ,ṭ,\eta) = \frac{2}{3}, \quad B_N(ṣ,ṭ,\zeta+\eta) = \frac{1}{3}$$

$$but, B_N(ṣ,ṛ,\zeta) \boxtimes B_N(ṛ,ṭ,\eta) \ge B_N(ṣ,ṭ,\zeta+\eta).$$

$Theorem$ 3.3. Every MS induces NMS.

$Proof$.

Let $(\Upsilon, \mathbb{d})$ be a metric space. Define $\tilde{\alpha} \boxtimes \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \tilde{\alpha} + \tilde{\eta} - \tilde{\alpha}\tilde{\eta}$, for all $\tilde{\alpha}, \tilde{\eta} \in [0,1]$ and $\forall\ ṣ,ṛ \in \Upsilon$, $and\ \zeta,\eta > 0$, we define $C_N, B_N, S_N : \Upsilon^2 \times \mathbb{R}^+ \longrightarrow [0,1]$ by

$$C_N(ṣ,ṛ,\zeta) = \frac{\zeta}{\zeta + \mathbb{d}(ṣ,ṛ)}, \quad B_N(ṣ,ṛ,\zeta) = \frac{\mathbb{d}(ṣ,ṛ)}{\zeta + \mathbb{d}(ṣ,ṛ)}, \quad S_N(ṣ,ṛ,\zeta) = \frac{\mathbb{d}(ṣ,ṛ)}{\zeta}.$$

To prove $(\Upsilon, \bar{\text{E}}, \boxtimes, \boxdot)$ is an NMS. Respect to C_N.

1) Since $\mathbb{d}(ṣ,ṛ) \ge 0 \Longrightarrow \zeta + \mathbb{d}(ṣ,ṛ) \ge \zeta \ \Longrightarrow\ 0 \le \dfrac{\zeta}{\zeta + \mathbb{d}(ṣ,ṛ)} \le 1 \Longrightarrow\ 0 \le C_N(ṣ,ṛ,\zeta) \le 1$.

2) Since, $\mathbb{d}(ṣ,ṛ) = 0\ iff\ ṣ = ṛ \ \Longrightarrow\ \zeta + \mathbb{d}(ṣ,ṛ) = \zeta\ iff\ ṣ = ṛ \Longrightarrow$

$$\frac{\zeta}{\zeta + \mathbb{d}(ṣ,ṛ)} = 1 \quad iff \quad ṣ = ṛ \quad \Longrightarrow \quad C_N(ṣ,ṛ,\zeta) = 1 \quad iff \quad ṣ = ṛ.$$

3) $C_N(ṣ,ṛ,\zeta) = \dfrac{\zeta}{\zeta + \mathbb{d}(ṣ,ṛ)} = \dfrac{\zeta}{\zeta + \mathbb{d}(ṛ,ṣ)} = C_N(ṛ,ṣ,\zeta)$.

4) $Since\ \dfrac{\zeta+\eta}{(\zeta+\eta) + \mathbb{d}(ṣ,ṛ)} - \dfrac{\zeta\eta}{\big(\zeta + \mathbb{d}(ṣ,ṭ)\big)\big(\eta + \mathbb{d}(ṭ,ṛ)\big)} \ge 0, then\ we\ have$

$$C_N(ṣ,ṛ,\zeta+\eta) \ge C_N(ṣ,ṭ,\zeta) \boxtimes C_N(ṭ,ṛ,\eta).$$

5) $Since\ \lim_{\zeta\to\infty} \dfrac{\zeta}{\zeta + \mathbb{d}(ṣ,ṛ)} = 1,\ then\ we\ have\ \lim_{\zeta\to\infty} C_N(ṣ,ṛ,\zeta) = 1$.

6) $Since\ C_N(ṣ,ṛ,.) : [0,\infty) \longrightarrow [0,1]\ is\ continuous$.

Similarly, all conditions for B_N, S_N can be verified. Therefore, $(\Upsilon, \bar{\text{E}}, \boxtimes, \boxdot)$ is an NMS induced by a metric $\mathbb{d}$, called the standard neutrosophic metric space.

Remark 3.4.
1) Can use other operations to prove the last theorem as (CTN) defined by $\tilde{\alpha} \boxtimes \tilde{\eta} = \min\{\tilde{\alpha}, \tilde{\eta}\}$ and (CTCN) define by $\tilde{\alpha} \boxdot \tilde{\eta} = \max\{\tilde{\alpha}, \tilde{\eta}\}$, or $\tilde{\alpha} \boxtimes \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \min\{\tilde{\alpha} + \tilde{\eta}, 1\}$
2) Reversal doesn't hold for Theorem 3.3; we'll show that in the next example.

Example 3.5. From example (3.1), since $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ is an NMS with operations $\tilde{\alpha} \boxtimes \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \tilde{\alpha} + \tilde{\eta} - \tilde{\alpha}\tilde{\eta}$, to show that, there exists a metric $\mathbb{d}$ on Υ satisfies for all $ṣ, ṛ \in \Upsilon$,

$$C_N(ṣ, ṛ, \zeta) = \frac{\zeta}{\zeta + \mathbb{d}(ṣ, ṛ)}, \quad B_N(ṣ, ṛ, \zeta) = \frac{\mathbb{d}(ṣ, ṛ)}{\zeta + \mathbb{d}(ṣ, ṛ)}, \quad S_N(ṣ, ṛ, \zeta) = \frac{\mathbb{d}(ṣ, ṛ)}{\zeta}., \quad \forall\, \zeta > 0$$

Let $\mathbb{d}$ be a metric on $\mathbb{X}$ that induces an NMS $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$.Then, $\forall\, \zeta > 0$, from C_N, that is,

$$C_N(ṣ, ṛ, \zeta) = \frac{\zeta}{\zeta + \mathbb{d}(ṣ, ṛ)} \implies \mathbb{d}(ṣ, ṛ) = \frac{\zeta[1 - C_N(ṣ, ṛ, \zeta)]}{C_N(ṣ, ṛ, \zeta)},$$

$$where, \quad C_N(ṣ, ṛ, \zeta) = \begin{cases} \dfrac{ṣ}{ṛ}, & ṣ \le ṛ \\[2ex] \dfrac{ṛ}{ṣ}, & ṛ \le ṣ \end{cases}$$

Take $ṣ = 1$, $ṛ = 2$, $ṭ = 3$, $\zeta = 2$. Then

$$C_N(ṣ, ṛ, \zeta) = \frac{1}{2}, \quad C_N(ṣ, ṭ, \zeta) = \frac{1}{3}, \quad C_N(ṛ, ṭ, \zeta) = \frac{2}{3}$$

Now,

$$\mathbb{d}(ṣ, ṛ) = \frac{2\left[1 - \frac{1}{2}\right]}{\frac{1}{2}} = 2, \quad \mathbb{d}(ṣ, ṭ) = \frac{2\left[1 - \frac{1}{3}\right]}{\frac{1}{3}} = 4, \quad \mathbb{d}(ṛ, ṭ) = \frac{2\left[1 - \frac{2}{3}\right]}{\frac{2}{3}} = 1.$$

Then we have $\mathbb{d}(ṣ, ṭ) > \mathbb{d}(ṣ, ṛ) + \mathbb{d}(ṛ, ṭ)$. This contradiction. Thus, $(\Upsilon, \mathbb{d})$ is not MS.

Example 3.6. Let $\Upsilon = \mathbb{R}$. Define $\tilde{\alpha} \boxtimes \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \tilde{\alpha} + \tilde{\eta} - \tilde{\alpha}\tilde{\eta}$ for each $\tilde{\alpha}, \tilde{\eta} \in [0,1]$ and define $C_N, B_N, S_N : \Upsilon^2 \times \mathbb{R}^+ \longrightarrow [0,1]$ by

$$C_N(ṣ, ṛ, \zeta) = exp^{\frac{-|ṣ-ṛ|}{\zeta}}, \quad B_N(ṣ, ṛ, \zeta) = 1 - exp^{\frac{-|ṣ-ṛ|}{\zeta}}, \quad S_N(ṣ, ṛ, \zeta) = 1 - exp^{\frac{-c|ṣ-ṛ|}{\zeta}}$$

for all $ṣ, ṛ \in \Upsilon$, and $\zeta, c > 0$. Then $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ is an NMS.

Example 3.7. Let $(\Upsilon = \mathbb{R}, \mathbb{d})$ be the usual metric space. Define $\tilde{\alpha} \boxtimes \tilde{\eta} = \min\{\tilde{\alpha}, \tilde{\eta}\}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \max\{\tilde{\alpha}, \tilde{\eta}\}$ for all $\tilde{\alpha}, \tilde{\eta} \in [0,1]$ and define $C_N, B_N, S_N : \Upsilon^2 \times \mathbb{R}^+ \longrightarrow [0,1]$by

$$C_N(ṣ, ṛ, \zeta) = \cos\left[\frac{\mathbb{d}(ṣ, ṛ))}{\zeta}\right], \quad B_N(ṣ, ṛ, \zeta) = 1 - \cos\left[\frac{\mathbb{d}(ṣ, ṛ)}{\zeta}\right], \quad S_N(ṣ, ṛ, \zeta) = \sin\left[\frac{\mathbb{d}(ṣ, ṛ)}{\zeta}\right],$$

for all $ṣ, ṛ \in \Upsilon$, and $\zeta > 0$. Then $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ is an NMS.

Theorem 3.8. Let $\mathbb{OB}(ṣ, \varrho_1, \zeta), \mathbb{OB}(ṣ, \varrho_2, \zeta)$ be two open balls in an NMS $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ with the same center $ṣ \in \Upsilon$. If $0 < \varrho_1 < \varrho_2 < 1$, then $\mathbb{OB}(ṣ, \varrho_1, \zeta) \subseteq \mathbb{OB}(ṣ, \varrho_2, \zeta)$.

Proof.
Let $ṛ \in \mathbb{OB}(ṣ, \varrho_1, \zeta) \implies C_N(ṣ, ṛ, \zeta) > 1 - \varrho_1, \; B_N(ṣ, ṛ, \zeta) < \varrho_1, \; S_N(ṣ, ṛ, \zeta) < \varrho_1$.
Since $0 < \varrho_1 < \varrho_2 < 1 \implies \varrho_1 < \varrho_2 \implies 1 - \varrho_1 > 1 - \varrho_2 \implies$
$C_N(ṣ, ṛ, \zeta) > 1 - \varrho_2, \; B_N(ṣ, ṛ, \zeta) < \varrho_2, \; S_N(ṣ, ṛ, \zeta) < \varrho_2 \implies ṛ \in \mathbb{OB}(ṣ, \varrho_2, \zeta) \implies$
$\mathbb{OB}(ṣ, \varrho_1, \zeta) \subseteq \mathbb{OB}(ṣ, \varrho_2, \zeta)$. Also, with respect to closed ball, i.e., $\mathbb{CB}[ṣ, \varrho_1, \zeta] \subseteq \mathbb{CB}[ṣ, \varrho_2, \zeta]$.

The next theorem improves upon the proof in [5] and was developed from intuitionistic fuzzy metric space.

Theorem 3.9. Open balls are an open set in an NMS.
Proof.
Consider an open ball $\mathbb{OB}(ṣ, ϱ, \zeta)$ in an NMS $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$, with center ṣ and radius ϱ.
Now,
let $ṛ \in \mathbb{OB}(ṣ, ϱ, \zeta)$, $\Rightarrow$ $C_N(ṣ, ṛ, \zeta) > 1 - ϱ$, $B_N(ṣ, ṛ, \zeta) < ϱ$, $S_N(ṣ, ṛ, \zeta) < ϱ$.
To show, ṛ is respect the center of open ball subset of the original ball $\mathbb{OB}(ṣ, ϱ, \zeta)$.
By using remark (2.6), we can find $\zeta_0, 0 < \zeta_0 < \zeta$, such that

$$C_N(ṣ, ṛ, \zeta_0) > 1 - ϱ, \quad B_N(ṣ, ṛ, \zeta_0) < ϱ, \quad S_N(ṣ, ṛ, \zeta_0) < ϱ$$

Now, respect to C_N.
Let $ϱ_0 = C_N(ṣ, ṛ, \zeta_0) > 1 - ϱ$, (1)
since $ϱ_0 > 1 - ϱ$, we can find $0 < \delta < 1$, such that
$ϱ_0 > 1 - \delta > 1 - ϱ$, (2)
for a given $ϱ_0$ and δ, such that $ϱ_0 > 1 - \delta$, we can find $0 < ϱ_1 < 1$, such that

$$ϱ_0 \boxtimes ϱ_1 \geq 1 - \delta. \quad (3)$$

Respect to B_N.
Let $1 - ϱ_0 = B_N(ṣ, ṛ, \zeta_0) < ϱ$, (4)
since $1 - ϱ_0 < ϱ$, we can find $0 < \delta < 1$, such that
$1 - ϱ_0 < \delta < ϱ$, (5)
for a given $ϱ_0$ and δ, such that $1 - ϱ_0 < \delta$, we can find $0 < ϱ_2 < 1$, such that
$(1 - ϱ_0) \boxdot (1 - ϱ_2) \leq \delta$. (6)
Respect to S_N.
Let $1 - ϱ_0 = S_N(ṣ, ṛ, \zeta_0) < ϱ$, (7)
since $1 - ϱ_0 < ϱ$, we can find $0 < \delta < 1$ such that
$1 - ϱ_0 < \delta < ϱ$, (8)
for a given $ϱ_0$ and δ, such that $1 - ϱ_0 < \delta$, we can find $0 < ϱ_3 < 1$, such that

$$(1 - ϱ_0) \boxdot (1 - ϱ_3) \leq \delta. \quad (9)$$

Put $ϱ_4 = max\{ϱ_1, ϱ_2, ϱ_3\}$.
Now consider the open ball $\mathbb{OB}(ṛ, 1 - ϱ_4, \zeta - \zeta_0)$. We claim
$\mathbb{OB}(ṛ, 1 - ϱ_4, \zeta - \zeta_0). \subseteq \mathbb{OB}(ṣ, ϱ, \zeta)$. To show that,
let $ṭ \in \mathbb{OB}(ṛ, 1 - ϱ_4, \zeta - \zeta_0)$, implies that

$$C_N(ṛ, ṭ, \zeta - \zeta_0) > ϱ_4, \quad B_N(ṛ, ṭ, \zeta - \zeta_0) < 1 - ϱ_4, \quad S_N(ṛ, ṭ, \zeta - \zeta_0) < 1 - ϱ_4. \quad (10)$$

Now, using (1) into (10) to get,

$$C_N(ṣ, ṭ, \zeta) = C_N(ṣ, ṭ, \zeta_0 + \zeta - \zeta_0) \geq C_N(ṣ, ṛ, \zeta_0) \boxtimes C_N(ṛ, ṭ, \zeta - \zeta_0)$$
$$\geq ϱ_0 \boxtimes ϱ_4 \geq ϱ_0 \boxtimes ϱ_1 \geq 1 - \delta > 1 - ϱ, \ \textit{that is} \ C_N(ṣ, ṭ, \zeta) > 1 - ϱ.$$

$$B_N(ṣ, ṭ, \zeta) = B_N(ṣ, ṭ, \zeta_0 + \zeta - \zeta_0) \leq B_N(ṣ, ṛ, \zeta_0) \boxdot B_N(ṛ, ṭ, \zeta - \zeta_0)$$
$$\leq (1 - ϱ_0) \boxdot (1 - ϱ_4) \leq (1 - ϱ_0) \boxdot (1 - ϱ_2) \leq \delta < ϱ, \ \textit{that is} \ B_N(ṣ, ṭ, \zeta) < ϱ.$$

Finally,

$$S_N(ṣ, ṭ, \zeta) = S_N(ṣ, ṭ, \zeta_0 + \zeta - \zeta_0) \leq S_N(ṣ, ṛ, \zeta_0) \boxdot S_N(ṛ, ṭ, \zeta - \zeta_0)$$
$$\leq (1 - ϱ_0) \boxdot (1 - ϱ_4) \leq (1 - ϱ_0) \boxdot (1 - ϱ_3) \leq \delta < ϱ, \ \textit{that is} \ S_N(ṣ, ṭ, \zeta) < ϱ.$$

Therefore $ṭ \in \mathbb{OB}(ṣ, ϱ, \zeta)$.
Theorem 3.10. Every closed ball in an NMS is a closed set.

Proof.
Consider the closed ball $\mathbb{CB}[ṣ, ϱ, \zeta]$ in an NMS $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$, with center ṣ and radius ϱ. Since

$$\mathbb{CB}[ṣ, ϱ, \zeta] \subseteq \overline{\mathbb{CB}[ṣ, ϱ, \zeta]}. \tag{1}$$

Now, to demonstrate the converse, let $ɧ \in \overline{\mathbb{CB}[ṣ, ϱ, \zeta]}$, then there exists a sequence $\{ɧ_m\}$ in $\mathbb{CB}[ṣ, ϱ, \zeta]$ such that $ɧ_m \longrightarrow ɧ$, as $m \longrightarrow \infty$.
Since $\{ɧ_m\}$ in $\mathbb{CB}[ṣ, ϱ, \zeta]$, implies that $C_N(ṣ, ɧ_m, \zeta) \geq 1 - ϱ, B_N(ṣ, ɧ_m, \zeta) \leq ϱ, S_N(ṣ, ɧ_m, \zeta) \leq ϱ$.
If $ɧ_m \longrightarrow ɧ$, then we have
$C_N(ɧ_m, ɧ, \zeta), B_N(ɧ_m, ɧ, \zeta)$ and $S_N(ɧ_m, ɧ, \zeta)$ are converges to $1, 0$ and 0 respectively, as $m \longrightarrow \infty$.
For a given $\eta > 0$,
$C_N(ṣ, ɧ, \zeta + \eta) \geq C_N(ṣ, ɧ_m, \zeta) \boxtimes C_N(ɧ_m, ɧ, \eta)$;
$B_N(ṣ, ɧ, \zeta + \eta) \leq B_N(ṣ, ɧ_m, \zeta) \boxdot B_N(ɧ_m, ɧ, \eta)$;
$S_N(ṣ, ɧ, \zeta + \eta) \leq S_N(ṣ, ɧ_m, \zeta) \boxdot S_N(ɧ_m, ɧ, \eta)$.
Taking a limit, then
$C_N(ṣ, ɧ, \zeta + \eta) \geq \lim_m C_N(ṣ, ɧ_m, \zeta) \boxtimes \lim_m C_N(ɧ_m, ɧ, \eta) \geq (1 - ϱ) \boxtimes 1 = 1 - ϱ$.
$B_N(ṣ, ɧ, \zeta + \eta) \leq \lim_m B_N(ṣ, ɧ_m, \zeta) \boxdot \lim_m B_N(ɧ_m, ɧ, \eta) \leq ϱ \boxdot 0 = ϱ$.
Finally,
$S_N(ṣ, ɧ, \zeta + \eta) \leq \lim_m S_N(ṣ, ɧ_m, \zeta) \boxdot \lim_m S_N(ɧ_m, ɧ, \eta) \leq ϱ \boxdot 0 = ϱ$.
In particular, for $m \in \mathbb{N}$, take $\zeta = \frac{1}{m}$. Then,

$$C_N\left(ṣ, ɧ, \zeta + \frac{1}{m}\right) \geq 1 - ϱ, \quad B_N\left(ṣ, ɧ, \zeta + \frac{1}{m}\right) \leq ϱ, \quad S_N\left(ṣ, ɧ, \zeta + \frac{1}{m}\right) \leq ϱ.$$

Hence,

$$C_N(ṣ, ɧ, \zeta) = \lim_m C_N\left(ṣ, ɧ, \zeta + \frac{1}{m}\right) \geq 1 - ϱ,$$

$$B_N(ṣ, ɧ, \zeta) = \lim_m B_N\left(ṣ, ɧ, \zeta + \frac{1}{m}\right) \leq ϱ,$$

$$S_N(ṣ, ɧ, \zeta) = \lim_m S_N\left(ṣ, ɧ, \zeta + \frac{1}{m}\right) \leq ϱ.$$

Therefore,

$$ɧ \in \mathbb{CB}[ṣ, ϱ, \zeta] \implies \overline{\mathbb{CB}[ṣ, ϱ, \zeta]} \subseteq \mathbb{CB}[ṣ, ϱ, \zeta] \tag{2}$$

From (1) and (2), we have the requirements.
Theorem 3.11. Let $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ be an NMS. Define $\mathfrak{I}_{(C,B,S)} = \{\hat{G} \subseteq \Upsilon : \hat{G} \text{ is open set}\}$, then $\mathfrak{I}_{(C,B,S)}$ is topology on Υ, called neutrosophic topology induced by an NMS. i.e., every NMS on Υ produces a topology $\mathfrak{I}_{(C,B,S)}$ on Υ. This relies on the collection of open sets of the form $\{\mathbb{OB}(ṣ, ϱ, \zeta) : ṣ \in \Upsilon, ϱ \in (0,1), \zeta > 0\}$.
Theorem 3.12 [5]. Every NMS is Hausdorff.
Theorem 3.13. Every NMS is normal.
Proof.
Let $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ be a neutrosophic metric space, and $\ddot{H}_1, \ddot{H}_2$ be mutually disjoint neutrosophic closed subset of Υ. Consider $ṣ \in \ddot{H}_1$ implies $ṣ \notin \ddot{H}_2$, then $ṣ \in \ddot{H}_2{}^c$(neutrosophic open subset). So, there exist $\zeta_ṣ > 0$ and $0 < ϱ_ṣ < 1$ such that $\mathbb{OB}(ṣ, ϱ_ṣ, \zeta_ṣ) \subseteq \ddot{H}_2{}^c$, i.e.$\mathbb{OB}(ṣ, ϱ_ṣ, \zeta_ṣ) \cap \ddot{H}_2 = \emptyset$.

Also, let $ṛ \in \ddot{H}_2$ implies $ṛ \notin \ddot{H}_1$, then $ṛ \in \ddot{H}_1{}^c$ (neutrosophic open subset), thus, there exist $\zeta_{ṛ} > 0$ and $0 < \varrho_{ṛ} < 1$ such that $\mathbb{OB}(ṛ, \varrho_{ṛ}, \zeta_{ṛ}) \subseteq \ddot{H}_1{}^c$, i.e, $\mathbb{OB}(ṛ, \varrho_{ṛ}, \zeta_{ṛ}) \cap \ddot{H}_1 = \emptyset$.
Let $\varrho = min\{\varrho_{ṣ}, \varrho_{ṛ}, \zeta_{ṣ}, \zeta_{ṛ}\}$, then we can choose $0 < \varrho_0 < \varrho$, such that
$(1-\varrho_0) \boxtimes (1-\varrho_0) > 1-\varrho$ and $\varrho_0 \boxdot \varrho_0 < \varrho$. Take

$$Ṵ = \bigcup_{ṣ \in \ddot{H}_1} \mathbb{OB}\left(ṣ, \varrho_0, \frac{\varrho}{2}\right) \quad , \quad \tilde{V} = \bigcup_{ṛ \in \ddot{H}_2} \mathbb{OB}\left(ṛ, \varrho_0, \frac{\varrho}{2}\right).$$

Then, $Ṵ$ and $\tilde{V}$ are open neutrosophic sets such that $\ddot{H}_1 \subseteq Ṵ$ and $\ddot{H}_2 \subseteq \tilde{V}$.
We assert that, $Ṵ \cap \tilde{V} = \emptyset$. Let $ṭ \in Ṵ \cap \tilde{V}$, then there exists $ṣ \in \ddot{H}_1$ and $ṛ \in \ddot{H}_2$ such that

$$ṭ \in \mathbb{OB}\left(ṣ, \varrho_0, \frac{\varrho}{2}\right), \ ṭ \in \mathbb{OB}\left(ṛ, \varrho_0, \frac{\varrho}{2}\right).$$

Thus, we have

$$C_N(ṣ, ṛ, \varrho) \geq C_N\left(ṣ, ṭ, \frac{\varrho}{2}\right) \boxtimes C_N\left(ṭ, ṛ, \frac{\varrho}{2}\right) \geq (1-\varrho_0) \boxtimes (1-\varrho_0) > 1-\varrho,$$

also,

$$B_N(ṣ, ṛ, \varrho) \leq B_N\left(ṣ, ṭ, \frac{\varrho}{2}\right) \boxdot B_N\left(ṭ, ṛ, \frac{\varrho}{2}\right) \leq \varrho_0 \boxdot \varrho_0 < \varrho$$

finally,

$$S_N(ṣ, ṛ, \varrho) \leq S_N\left(ṣ, ṭ, \frac{\varrho}{2}\right) \boxdot S_N\left(ṭ, ṛ, \frac{\varrho}{2}\right) \leq \varrho_0 \boxdot \varrho_0 < \varrho.$$

Hence, $ṛ \in \mathbb{OB}(ṣ, \varrho, \varrho)$, but $\varrho < \varrho_{ṣ}, \zeta_{ṣ}$ (its minimum) therefore $\mathbb{OB}(ṣ, \varrho, \varrho) \subseteq \mathbb{OB}(ṣ, \varrho_{ṣ}, \zeta_{ṣ})$, and thus $\mathbb{OB}(ṣ, \varrho_{ṣ}, \zeta_{ṣ}) \cap \ddot{H}_2 \neq \emptyset$, which is an inconsistency. Accordingly, Υ is normal.
Corollary 3.14. Every NMS is $T_4 - space$.
$Proof$.
Let $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ be a neutrosophic metric space. Since an NMS is Normal and Hausdorff, so it's $T_4 - space$.
$Theorem$ 3.15. Any compact NMS is separable.
$Proof$.
Consider $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ be compact NMS. Let $0 < \varrho < 1$ and $\zeta > 0$. Since Υ is compact, thus there exist $ṣ_1, ṣ_2, \dots, ṣ_{ḿ}$ in Υ such that

$$\Upsilon = \bigcup_{\ell=1}^{ḿ} \mathbb{OB}(ṣ_{\ell}, \varrho, \zeta).$$

Particularly, for each $ḿ \in \mathbb{N}$, it is possible to pick a finite subset $D_{ḿ}$ such that

$$\Upsilon = \bigcup_{d \in D_{ḿ}} \mathbb{OB}\left(d, \frac{1}{ḿ}, \frac{1}{ḿ}\right).$$

Let $D = \bigcup_{ḿ \in \mathbb{N}} D_{ḿ}$, then D is countable. We contend that $\Upsilon \subseteq \overline{D}$, to show that, let $ṣ \in \Upsilon$, subsequently, for each $ḿ \in \mathbb{N}$, at least one $d_{ḿ} \in D_{ḿ}$ such that $ṣ \in \mathbb{OB}\left(d_{ḿ}, \frac{1}{ḿ}, \frac{1}{ḿ}\right)$. Thus $d_{ḿ}$ converges to $ṣ$, but $d_{ḿ} \in D$ for all $ḿ$, thus $ṣ \in \overline{D}$. Since $\overline{D} \subseteq \Upsilon$, we obtain D is dense in Υ and thus Υ is separable.
$Theorem$ 3.17. Uniqueness of the limit holds for convergent sequences in an NMS.
$Proof$.
Consider $\{ϟ_n\}$ a convergent sequence in an NMS $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$.

If possible, $\lim_{n\to\infty} ϟ_n = ϟ$ and $\lim_{n\to\infty} ϟ_n = ϱ$ with $ϟ \neq ϱ$. For each $\zeta,\eta > 0$, then we have

$$\lim_{n\to\infty} C_N(ϟ_n, ϟ, \zeta) = 1 \quad , \quad \lim_{n\to\infty} B_N(ϟ_n, ϟ, \zeta) = 0 \quad , \quad \lim_{n\to\infty} S_N(ϟ_n, ϟ, \zeta) = 0.$$

And

$$\lim_{n\to\infty} C_N(ϟ_n, ϱ, \eta) = 1 \quad , \quad \lim_{n\to\infty} B_N(ϟ_n, ϱ, \eta) = 0 \quad , \quad \lim_{n\to\infty} S_N(ϟ_n, ϱ, \eta) = 0.$$

Now,

$C_N(ϟ, ϱ, \zeta+\eta) \geq C_N(ϟ, ϟ_n, \zeta) \boxtimes C_N(ϟ_n, ϱ, \eta) = C_N(ϟ_n, ϟ, \zeta) \boxtimes C_N(ϟ_n, ϱ, \eta)$,

taking limit as $n \longrightarrow \infty$, and for $\zeta,\eta \longrightarrow \infty$,

$$C_N(ϟ, ϱ, \zeta+\eta) \geq 1 \boxtimes 1 = 1.$$

Also,

$B_N(ϟ, ϱ, \zeta+\eta) \leq B_N(ϟ, ϟ_n, \zeta) \boxdot B_N(ϟ_n, ϱ, \eta) = B_N(ϟ_n, ϟ, \zeta) \boxdot B_N(ϟ_n, ϱ, \eta)$,

taking limit as $n \longrightarrow \infty$, and for each $\zeta,\eta \longrightarrow \infty$,

$$B_N(ϟ, ϱ, \zeta+\eta) \leq 0 \boxdot 0 = 0.$$

Similarly, $S_N(ϟ, ϱ, \zeta+\eta) = 0$, by definition of a neutrosophic metric space, we obtain $ϟ = ϱ$.

Theorem 3.18. Every convergent sequence in an NMS is a Cauchy.

Proof.

Let $\{ϟ_n\}$ be a convergent sequence in an NMS $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$, with $\lim_{n\to\infty} ϟ_n = ϟ$.

Then, for $\zeta > 0$, there is $n_0 \in \mathbb{N}$, getting,

$$\lim_{n\to\infty} C_N(ϟ_n, ϟ, \zeta) = 1 \quad , \quad \lim_{n\to\infty} B_N(ϟ_n, ϟ, \zeta) = 0 \quad , \quad \lim_{n\to\infty} S_N(ϟ_n, ϟ, \zeta) = 0.$$

For each $n, m \geq n_0$,

$$C_N(ϟ_n, ϟ_m, \zeta) \geq C_N\left(ϟ_n, ϟ, \frac{\zeta}{2}\right) \boxtimes C_N\left(ϟ, ϟ_m, \frac{\zeta}{2}\right) = C_N\left(ϟ_n, ϟ, \frac{\zeta}{2}\right) \boxtimes C_N\left(ϟ_m, ϟ, \frac{\zeta}{2}\right).$$

Taking, limit, then

$$\lim_{n,m\to\infty} C_N(ϟ_n, ϟ_m, \zeta) = 1 \boxtimes 1 = 1.$$

Also,

$$B_N(ϟ_n, ϟ_m, \zeta) \leq B_N\left(ϟ_n, ϟ, \frac{\zeta}{2}\right) \boxdot B_N\left(ϟ, ϟ_m, \frac{\zeta}{2}\right) = B_N\left(ϟ_n, ϟ, \frac{\zeta}{2}\right) \boxdot B_N\left(ϟ_m, ϟ, \frac{\zeta}{2}\right).$$

Taking, limit, then

$$\lim_{n,m\to\infty} B_N(ϟ_n, ϟ_m, \zeta) = 0 \boxdot 0 = 0.$$

Similarly,

$$\lim_{n,m\to\infty} S_N(ϟ_n, ϟ_m, \zeta) = 0.$$

Therefore, the sequence $\{ϟ_n\}$ is a Cauchy sequence. In general, the converse of Theorem 3.18 fails. A counterexample is given in the next example.

Example 3.19. Let $(\Upsilon = \mathbb{R} \setminus \{0\}, \mathbb{d})$ be a usual metric space. Define $\tilde{\alpha} \boxtimes \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \tilde{\alpha} + \tilde{\eta} - \tilde{\alpha}\tilde{\eta}$ for each $\tilde{\alpha}, \tilde{\eta} \in [0,1]$ and define $C_N, B_N, S_N : \Upsilon^2 \times \mathbb{R}^+ \longrightarrow [0,1]$ by

$$C_N(ṣ, ṛ, \zeta) = \frac{\zeta}{\zeta + \mathbb{d}(ṣ, ṛ)}, \quad B_N(ṣ, ṛ, \zeta) = \frac{\mathbb{d}(ṣ, ṛ)}{\zeta + \mathbb{d}(ṣ, ṛ)}, \quad S_N(ṣ, ṛ, \zeta) = \frac{\mathbb{d}(ṣ, ṛ)}{\zeta}, \quad ṣ, ṛ \in \Upsilon, \forall \zeta > 0.$$

Then, $(\Upsilon, \bar{E}, \boxtimes, \boxdot)$ is an NMS induced by $\mathbb{d}$. Take the sequence $ϟ_{\wp} = \left\{\frac{1}{\wp}, \forall \wp \in \mathbb{N}\right\}$. Since

$$\lim_{\wp,q\to\infty} C_N(ϟ_{\wp}, ϟ_q, \zeta) = \lim_{\wp,q\to\infty} \frac{\zeta}{\zeta + \left|\frac{1}{\wp} - \frac{1}{q}\right|} = 1.$$

$$\lim_{\wp,\mathscr{q}\to\infty} B_N(\natural_\wp,\natural_\mathscr{q},\zeta) = \lim_{\wp,\mathscr{q}\to\infty} \frac{\left|\frac{1}{\wp}-\frac{1}{\mathscr{q}}\right|}{\zeta+\left|\frac{1}{\wp}-\frac{1}{\mathscr{q}}\right|} = 0.$$

$$\lim_{\wp,\mathscr{q}\to\infty} S_N(\natural_\wp,\natural_\mathscr{q},\zeta) = \lim_{\wp,\mathscr{q}\to\infty} \frac{\left|\frac{1}{\wp}-\frac{1}{\mathscr{q}}\right|}{\zeta} = 0.$$

Thus, $\{\natural_n\}$ is Cauchy sequence, but is not convergent sequence in Υ, since $\frac{1}{\wp}\longrightarrow 0 \notin \Upsilon$.

Example 3.20. Let $(\Upsilon=\mathbb{R},\mathbb{d})$ be a usual metric space. Define $\tilde{\alpha} \boxtimes \tilde{\eta} = \tilde{\alpha}\tilde{\eta}$ and $\tilde{\alpha} \boxdot \tilde{\eta} = \tilde{\alpha}+\tilde{\eta}-\tilde{\alpha}\tilde{\eta}$ for each $\tilde{\alpha},\tilde{\eta}\in[0,1]$ and define $C_N, B_N, S_N : \Upsilon^2\times\mathbb{R}^+\longrightarrow[0,1]$ by

$$C_N(ṣ,ṛ,\zeta)=\frac{\zeta}{\zeta+\mathbb{d}(ṣ,ṛ)},\quad B_N(ṣ,ṛ,\zeta)=\frac{\mathbb{d}(ṣ,ṛ)}{\zeta+\mathbb{d}(ṣ,ṛ)},\quad S_N(ṣ,ṛ,\zeta)=\frac{\mathbb{d}(ṣ,ṛ)}{\zeta},\quad ṣ,ṛ\in\mathbb{X},\forall\,\zeta>0.$$

Then, $(\Upsilon,\bar{E},\boxtimes,\boxdot)$ is a complete NMS, it's clear that.

Conclusions:

This paper builds on the concept of an NMS and discuss some properties such as open, closed ball, normalness, compactness, the convergence of sequence, characteristics of Cauchy sequence and achieve their relationship within an NMS.

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