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# Representations and Dual Structures of $\Gamma$ −Lie Color Algebras

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**ABSTRACT**

In this paper, we introduced new concepts which are $\Gamma$ −representation , dual $\Gamma$ − representation, and the dual adjoint $\Gamma$ −representation and study them on $\Gamma$ − Lie color algebra . And prove that the adjoint gives a natural Γ −representation. Moreover, for any Γ −representation, we construct the associated semidirect product and show that it carries a $\Gamma - Lie$ color algebra structure.

## 1. Introduction

In [4], F. B. Gonzalez,introduced Lie algebras as one of the most important algebraic structures in mathematics. These algebras were originally developed to study continuous symmetries and have become fundamental tools in mathematics, geometry, and physics. In [8], by V. G. Kac, introduced Lie super algebras in 1977as a generalization of Lie algebras by the development of mathematics and the emergence of concepts such as super symmetry .In [7],by M. Scheunert, introduced Lie color algebras specifically as structures built upon an arbitrary abelian grading group equipped with a bicharacter. in [6], by J. Li and L. Chen, studied the theory of a representation of 3-Bi-Hom Lie algebra inserted. in [5], by I. Laraiedh, defined a representation of n- Bi-Hom Lie algebras and obtained the semi direct product n-Bi-Hom Lie algebra associated with any representation ρ of an n-Bi-Hom Lie algebra $G$ on a vector space $V$ . in [2], by A. H. Rezaei and B. Davvaz, introduced the concept of Gamma algebra and Gamma Lie algebra as new algebra structures and discussed Some of their fundamental concept and properties. In [1] by A. Alzaiad and R. Shaheen, study on involutive Gamma derivations on 3-Pre Gamma-Lie algebra and n-Gamma Lie algebra. In [3] by Abdulhasan and R. Shaheen, study on generalized Γ- Lie triple derivations of Γ- Lie color algebras and their sub algebras. In this paper we introduced a new concept which are Γ- representation of Γ-Lie color algebra and equivalent two representations of an Γ- Lie color algebra and give some results about them, we also study , dual $\Gamma$ − representation, and the dual adjoint $\Gamma$ − representation on $\Gamma$ − Lie color algebra now, we shall remember some basic definitions and fundamental concepts.

## 2. Preliminaries

**Definition 2.1**:-**[7]** Suppose an abelian group G and a vector space V over a field F, V is called a G-graded vector space if there exists a family of vector subspaces $\{V_g\}_{g\in G}$ such that $V = \oplus_{g\in G} V_g$. That is, An element $x \in V$ is called homogeneous of degree g if $x \in V_g$ In this case, the degree of x is denoted by $|x| = g$. The set of all homogeneous elements of V is denoted by hg(V).

**Definition 2.2:-[7]** Consider two G-graded vector spaces V and W. A linear map $H: V \to W$ is classified as homogeneous of degree $g \in G_g$ provided that $H(V_h) \subseteq W_{g+h}$ for all $h \in G$.

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**Definition2.3:-[7]** Let G be an abelian group and let F be a field.A map $\varepsilon : G \times G \to F\backslash\{0\}$ is called a skew − symmetric bicharacter if, for all $a, b, c \in G$,

1) $\varepsilon(a,b)\,\varepsilon(b,a) = 1$,
2) $\varepsilon(a,b+c) = \varepsilon(a,b)\,\varepsilon(a,c)$,
3) $\varepsilon(a+b,c) = \varepsilon(a,c)\,\varepsilon(b,c)$.

**Definition 2.4:-[7]** Let G be an abelian group, F be a field, and let $\varepsilon : G \times G \to F\backslash\{0\}$ be a skew-symmetric bicharacter. A G-graded vector space $L = \bigoplus_{a\in G} L_a$ over F, together with a graded bilinear bracket$[\,,] : L \times L \to L$,is said to be a Lie color algebra if, for all homogeneous elements $x \in L_a, y \in L_b, z \in L_c$, where $a, b, c \in G$, $L_a, L_b, L_c \subseteq L$ ,the following identities hold:

1) Grading condition $[L_a, L_b] \subseteq L_{a+b}$
2) $\varepsilon$ − skew-symmetry $[x,y] = -\varepsilon(a,b)[y,x]$.
3) $\varepsilon$-Jacobi identity $\varepsilon(c,a)[s,[y,z]] + \varepsilon(a,b)[t,[z,x]] + \varepsilon(b,c)[z,[x,y]] = 0$.

**Definition 2.5:-[2]** Let $\Gamma$ be a grouboid and V be a vector space over afield F . then V is called a $\Gamma$ − algebra defined over the field F if there exists a mapping : $v \times \Gamma \times v \to v$ ( the image is written as $s\lambda t$, for $s,t,u \in v$ and $\lambda \in \Gamma$ ) satisfiying the following conditions:

1) $(s+t)\lambda u = s\lambda u + t\lambda u$
2) $s(\lambda+\beta)t = s\lambda t + s\beta t$
3) $(cs)\lambda t = c(s\lambda t) = s\lambda(ct)\ \forall s,t,u \in V, C \in F, \lambda,\beta \in \Gamma$, furthermore, $\Gamma$ − algebra is said to be associative if
4) $(s\lambda t)\beta u = s\lambda(t\beta u)$ .

**Definition2.6:-[2]** Let V is an associative $\Gamma$-algebra defind on a field F. Then, for any $\lambda \in \Gamma$ one can construct an $\Gamma$-Lie algebra $L_\lambda.(V)$ As a$\Gamma$ −vector space $L_\lambda.(V)$ is equal to V. The Lie $\Gamma$- bracket of two elements of$L_\lambda.(V)$ is defined to be their commutator in V, $[u,v]_\lambda = u\lambda v - v\lambda u$. Note that $[u,v]_\lambda = -[v,u]_\lambda$.

**Definition 2.7 :-[3]** Fix an abelian group G, a field F, and a skew-symmetric bicharacter let $\varepsilon : G \times G \longrightarrow F\backslash\{0\}$. A G-graded vector space $L = \bigoplus_{a\in G} L_a$ over F, equipped with a graded bilinear bracket $[\,,]_\lambda : L \times \Gamma \times L \to L$, is called a $\Gamma$- Lie color algebra provided that, for homogeneous elements $x \in L_a$, $y \in L_b$, $z \in L_c$ with $a,b,c \in G$ , $L_a, L_b, L_c \subseteq L$ and $\lambda \in \Gamma$, the conditions below are satisfied:

1) Grading condition $[L_a, L_b]_\lambda \subseteq L_{a+b}$
2) $\varepsilon$-skew-symmetry $[x,y]_\lambda = -\varepsilon(a,b)[y,x]_\lambda$.
3) $\varepsilon$-Jacobi identity $\varepsilon(c,a)[x,[y,z]_\lambda]_\lambda + \varepsilon(a,b)[t,[z,x]_\lambda]_\lambda + \varepsilon(b,c)[u,[x,y]_\lambda]_\lambda = 0$.

## 3. Main Results

**Definition 3.1:-** Let a $\Gamma$ −Lie color algebra $L$ together with a G-graded vector space V. a $\Gamma$ − representation of L on V is a linear map $p: L \to End(V)$ satisfying

$$p([x,y]_\lambda) = p(x) \circ p(y) - \varepsilon(a,b)\, p(y) \circ p(x)$$

for all homogeneous elements $x \in L_a, y \in L_b, \lambda \in \Gamma$,The pair $(V, p)$ is called a $\Gamma-$ representation of the $\Gamma-$ Lie color algebra L.

**Definition 3.2:-** Suppose L is a $\Gamma-$ Lie color algebra and let $(V, p)$ be a $\Gamma-$ representation of L.The semidirect product of L and V is the $G$-graded vector space $L \ltimes V$ equipped with the bracket

$$[(x,u),(y,v)]_\lambda = \big([x,y]_\lambda, p(x)(v) - \varepsilon(a,b)p(y)(u)\big),$$

for all homogeneous elements $x, y \in hg(L)$ and $u, v \in hg(V)$.The graded vector space $L \ltimes V$ equipped with the above bracket is called the semidirect product of L and V associated with the $\Gamma-$ representation p.

**Proposition 3.3:-** Given a $\Gamma-$Lie color algebra L. Define $ad: L \to End(L)$ by $ad_x(y) = [x,y]_\lambda$ for all homogeneous elements $x \in L_a, y \in L_b, \lambda \in \Gamma$.Then $(L, ad)$ is a $\Gamma-$ representation of the $\Gamma-$Lie color algebra L, called the adjoint $\Gamma-$ representation.

**Proof**:- For all homogeneous elements $x \in L_a, y \in L_b, z \in L_c, \lambda \in \Gamma$ we have $ad_{[x,y]_\lambda}(z) = [[x,y]_\lambda, z]_\lambda$ In a similar manner, $ad_x \text{ o } ad_y(z) = ad_x([y,z]_\lambda) = [x,[y,z]_\lambda]_\lambda$,and $ad_y \text{ o } ad_x(z) = [y,[x,z]_\lambda]_\lambda$ As a result, $ad_x \text{ o } ad_y(z) - \varepsilon(a,b) ad_y \text{ o } ad_x(z) = [x,[y,z]_\lambda]_\lambda - \varepsilon(a,b)[y,[x,z]_\lambda]_\lambda$. Using the ε-Jacobi identity, $\varepsilon(c,a)[x,[y,z]_\lambda]_\lambda + \varepsilon(a,b)[y,[z,x]_\lambda]_\lambda + \varepsilon(b,c)[z,[x,y]_\lambda]_\lambda = 0$, and the ε-skew symmetry $[z,x]_\lambda = -\varepsilon(c,a)[x,z]_\lambda$, we arrive at $[x,[y,z]_\lambda]_\lambda - \varepsilon(a,b)[y,[x,z]_\lambda]_\lambda = [[x,y]_\lambda, z]_\lambda$. Thus we conclude that, $ad_{[x,y]_\lambda} = ad_x \text{ o } ad_y - \varepsilon(a,b) ad_y \text{ o } ad_x$. Thus, ad satisfies the $\Gamma-$ representation condition, and consequently $(L, ad))$ is a $\Gamma-$ representation of L. ■

**Proposition 3.4:-** Take $L$ to be a Γ−Lie color algebra equipped with $(V, p)$ a $\Gamma-$ representation of L $(V, p)$. On the graded vector space $L \oplus V$ introduce the bracket operation

$$[(x,u),(y,v)]_\lambda = ([x,y]_\lambda, p(x)v - \varepsilon(a,b)p(y)u),$$

for homogeneous $x \in L_a, y \in L_b, \lambda \in \Gamma$ and $u, v \in V$. Then $L \oplus V$ is a $\Gamma-$Lie color algebra, called the semi-direct product of L and V, and denoted by $L \ltimes V$.

**Proof:-** It remains to verify that $[\cdot,\cdot]_\lambda$ conforms to each axiom of a $\Gamma-$Lie color algebra.

(i) Grading Condition

Let $x \in L_a, y \in L_b, \lambda \in \Gamma$, and $u \in V_a, v \in V_b$. Since L is a $\Gamma-$Lie color algebra,$[L_a, L_b]_\lambda \subseteq L_{\lambda(a+b)}$. As a result, $[x,y]_\lambda \in L_{\lambda(a+b)}$. Moreover, since $p(x)$ is homogeneous of degree $a$, $p(x)v \in V_{(a+b)}$, and by an analogous argument, $p(y)u \in V_{(a+b)}$. Thus we conclude that,

$$[(x,u),(y,v)]_\lambda \in (L \oplus V)_{(a+b)}.$$

Thus,

$[(L \oplus V)_a, (L \oplus V)_b]_\lambda \subseteq (L \oplus V)_{(a+b)}$, and the grading condition holds.

(ii) ε-Skew Symmetry

We compute $[(y,v),(x,u)]_\lambda = ([y,x]_\lambda, \rho(y)u - \varepsilon(b,a)\, \rho(x)v)$. Using the ε-skew symmetry in L, $[y,x]_\lambda = -\varepsilon(b,a)[x,y]_\lambda$, and the relation $\varepsilon(a,b)\varepsilon(b,a) = 1$, we arrive at

$$-\varepsilon(a,b)[(y,v),(x,u)]_\lambda = ([x,y]_\lambda, \rho(x)v - \varepsilon(a,b)\, \rho(y)u).$$

As a result,

$$[(x,u),(y,v)]_\lambda = -\varepsilon(a,b)[(y,v),(x,u)]_\lambda .$$

Thus we conclude that, the bracket is ε-skew symmetric.

(iii) ε-Jacobi Identity

Let $(x,u),(y,v),(z,w) \in L \oplus V$ be homogeneous elements we verify that

$$\varepsilon(c,a)[(x,u),[(y,v),(z,w)]_\lambda]_\lambda + \varepsilon(a,b)[(y,v),[(z,w),(x,u)]_\lambda]_\lambda$$

$$+\varepsilon(b,c)[(z,w),[(x,u),(y,v)]_\lambda]_\lambda = 0$$

First, $[(y,v),(z,w)]_\lambda = ([y,z]_\lambda, \rho(y)w - \varepsilon(b,c)\, \rho(z)v)$. As a result,

$$[(x,u),[(y,v),(z,w)]_\lambda]_\lambda = [(x,u),([y,z]_\lambda, \rho(y)w - \varepsilon(b,c)\, \rho(z)v)]_\lambda$$

$$= ([x,[y,z]_\lambda]_\lambda, \rho(x) \circ \rho(y)w - \varepsilon(b,c)\, \rho(x) \circ \rho(z)v - \varepsilon(a,b+c)\, \rho([y,z]_\lambda)u).$$

By an analogous argument,

$$[(y,v),[(z,w),(x,u)]_{\lambda\, p}]_\lambda =$$

$$([y,[z,x]_\lambda]_\lambda, \rho(y) \circ \rho(z)u - \varepsilon(c,a)\, \rho(y) \circ \rho(x)w - \varepsilon(b,c+a)\, \rho([z,x]_\lambda)v),$$

and $[(z,w),[(x,u),(y,v)]_\lambda]_\lambda =$

$$([z,[x,y]_\lambda]_\lambda, \rho(z) \circ \rho(x)v - \varepsilon(a,b)\, \rho(z) \circ \rho(y)u - \varepsilon(c,a+b)\rho([x,y]_\lambda)w).$$

Thus we conclude that,

$$\varepsilon(c,a)[(x,u),[(y,v),(z,w)]_\lambda]_\lambda + \varepsilon(a,b)[(y,v),[(z,w),(x,u)]_\lambda]_\lambda + \varepsilon(b,c)[(z,w),[(x,u),(y,v)]_\lambda]_\lambda = (A,B),$$

where

$A = \varepsilon(c,a)[x,[y,z]_\lambda]_\lambda + \varepsilon(a,b)[y,[z,x]_\lambda]_\lambda + \varepsilon(b,c)[z,[x,y]_\lambda]_\lambda$, and

$B = \varepsilon(c,a)(\,p(x)\ o\ \ p(y)w - \varepsilon(b,c)\,p(x)\ o\ \ p(z)v - \varepsilon(a,b+c)\,p([y,z]_\lambda)u)$

$+\varepsilon(a,b)(\,p(y)\ o\ \ p(z)u - \varepsilon(c,a)\,p(y)\ o\ \ p(x)w - \varepsilon(b,c+a)\,p([z,x]_\lambda)v)$

$+\varepsilon(b,c)(\,p(z)\ o\ \ p(x)v - \varepsilon(x,y)\,p(z)\ o\ \ p(y)u - \varepsilon(c,a+b)\,p([x,y]_\lambda)w)$.

Since $L$ is a $\Gamma$ −Lie color algebra, $A = 0$

$$\varepsilon(c,a)[x,[y,z]_\lambda]_\lambda + \varepsilon(a,b)[y,[z,x]_\lambda]_\lambda + \varepsilon(b,c)[z,[x,y]_\lambda]_\lambda, = 0$$

by the ε-Jacobi identity. Now we simplify B. Coefficients of $w$ The terms containing $w$ are

$\varepsilon(c,a)\,p(x)\ o\ \ p(y)w - \varepsilon(a,b)\varepsilon(c,a)\,p(y)\ o\ \ p(x)w - \varepsilon(b,c)\varepsilon(c,a+b)\,p([x,y]_\lambda)w$.

Using $\varepsilon(c,a+b) = \varepsilon(c,a)\varepsilon(c,b)$,and $\varepsilon(b,c)\varepsilon(c,b) = 1$, we arrive at

$\varepsilon(b,c)\varepsilon(c,a+b) = \varepsilon(c,a)$. As a result,

$= \varepsilon(c,a)(\,p(x)\ o\ \ p(y) - \varepsilon(a,b)\,p(y)\ o\ \ p(x) - \ p([x,y]_\lambda))w$.

Since $(V,p)$ is a $\Gamma$ − representation,

$$p([x,y]_\lambda) = \ p(x)\ o\ \ p(y) - \varepsilon(a,b)\,p(y)\ o\ \ p(x),$$

Thus we conclude that the coefficient of $w$ vanishes.Using the $\Gamma$ − representation once again, the coefficient of $v,u$ vanishes

Similary ,the coefficient of $u,v$ also vanish ,Thus we conclude that, $B = 0$ Consequently,

$\varepsilon(c,a)[(x,u),[(y,v),(z,w)]_\lambda]_\lambda + \varepsilon(a,b)[(y,v),[(z,w),(x,u)]_\lambda]_\lambda$

$+\varepsilon(b,c)[(z,w),[(x,u),(y,v)]_\lambda]_\lambda = 0$.

Thus, the bracket satisfies the $\varepsilon$ − Jacobi identity,Thus we conclude that, $L \oplus V$is a $\Gamma$ − Lie color algebra. ■

**Definition 3.5:-** Given two $\Gamma$-representation $(V_1,\ p_1)$and$(V_2,\ p_2)$ of a $\Gamma$ −Lie color algebra L, we call them equivalent whenever a graded vector space isomorphism $T: V_1 \to V_2$ satisfies $T \circ p_1(x) = p_2(x) \circ T$ for every homogeneous $x \in L$, equivalently, $T(\,p_1(x)(u)) = \ p_2(x)(T(u)), \forall x \in L,\ u \in V_1$.

**Theorem 3.6:-** Suppose L is a $\Gamma$ −Lie color algebra carrying a $\Gamma$-representation $(V, p)$, and let $\alpha: L \to L$ be an even $\Gamma$ −Lie color algebra endomorphism obeying $\alpha([x, y]_\lambda) = [\alpha(x), \alpha(y)]_\lambda$ for homogeneous $x \in L_a, y \in L_b, \lambda \in \Gamma$. Introduce the linear map $\breve{p}: L \to End(V)$ via $\breve{p}(x) = p(\alpha(x)), \forall\, x \in L$. where , $\breve{p} = p \circ \alpha$. Then $(V, \breve{p})$ is a $\Gamma$-representation of the $\Gamma$ −Lie color algebra L.

**Proof:-** Let $x \in L_a, y \in L_b, \lambda \in \Gamma$, be homogeneous elements. Recalling the definition of $\breve{p}$ ,

$\breve{p}([x, y]_\lambda) = p(\alpha([x, y]_\lambda))$.Since $\alpha$ is a $\Gamma$ − Lie color algebra endomorphism,

$$\alpha([x, y]_\lambda) = [\alpha(x), \alpha(y)]_\lambda.$$

As a result,

$$\breve{p}([x, y]_\lambda) = p([\alpha(x), \alpha(y)]_\lambda).$$

Because $(V, p)$ is a representation of L,

$$p([\alpha(x), \alpha(y)]_\lambda) = p(\alpha(x)) \circ p(\alpha(y)) - \varepsilon(\alpha(x), \alpha(y))\, p(\alpha(y)) \circ p(\alpha(x)).$$

Since $\alpha$ is even,$\alpha(x) = x, \alpha(y) = y$,Thus we conclude that $\varepsilon(\alpha(x), \alpha(y)) = \varepsilon(x, y)$.Thus,

$$\breve{p}([x, y]_\lambda) = p(\alpha(x)) \circ p(\alpha(y)) - \varepsilon(a, b)\, p(\alpha(y)) \circ p(\alpha(x)).$$

Applying the formulation of $\breve{p}$

$$\breve{p}(x) = p(\alpha(x)), \qquad \breve{p}(y) = p(\alpha(y)),$$

we arrive at

$$\breve{p}([x, y]_\lambda) = \breve{p}(x) \circ \breve{p}(y) - \varepsilon(a, b)\, \breve{p}(y) \circ \breve{p}(x).$$

Thus we conclude that, $(V, \breve{p})$ is a $\Gamma$ − representation of L.

**Definition 3.7:-** Take $L$ to be a $\Gamma$ −Lie color algebra admitting a $\Gamma$-representation $(V, p)$, and let $V^* = Hom(V, F)$ denote the dual space of V. For homogeneous $x \in L_a$ and $f \in (V^*)$,($deg(f) = e \in G$) define a linear map $p^*: L \to End(V^*)$ through $(p^*(x)(f))(u) = -\varepsilon(a, e)\, f(p(x)(u)), \forall u \in V$. This map $p^*$ is termed the dual $\Gamma$ − representation associated with $p$.

**Theorem 3.8:-** Given a $\Gamma$-Lie color algebra $L$ with $\Gamma$-representation $(V, p)$, define $p^*: L \to End(V^*)$ via $p^*(x)(f))(u) = -\varepsilon(a, e)\, f(p(x)(u))$, for all $x \in hg(L_a), f \in hg(V^*), (deg(f) = e \in G), u \in V$. Then $(V^*, p^*)$ is a $\Gamma$ − representation of the $\Gamma$ −Lie color algebra L.

**Proof:-** Let $x \in hg(L_a), y \in hg(L_b), f \in hg(V^*), u \in V$. we verify that $p^*$ satisfies the $\Gamma -$ representation

$$p^*([x,y]_\lambda) = p^*(x) \text{ o } p^*(y) - \varepsilon(a,b)\, p^*(y) \text{ o } p^*(x).$$

Recalling the definition of $p^*$,

$$(p^*([x,y]_\lambda)(f))(u) = -\varepsilon([x,y]_\lambda, e)\, f(p([x,y]_\lambda)(u)).$$

Since the bracket is graded, $|\, [x,y]_\lambda \,| = a + b$,

Thus we conclude that $\varepsilon([x,y]_\lambda, e) = \varepsilon(a+b, e)$. Using the bicharacter property, $\varepsilon(a+b,e) = \varepsilon(a,e)\varepsilon(b,e)$.

As a result, $(p^*([x,y]_\lambda)(f))(u) = -\varepsilon(a,e)\varepsilon(b,e)f(p([x,y]_\lambda)(u))$.

Because $(V,p)$ is a $\Gamma$-representation of L, $p([x,y]_\lambda) = p(x) \text{ o } p(y) - \varepsilon(a,b)\, p(y) \text{ o } p(x)$.

Substituting this equality, we arrive at

$(p^*([x,y]_\lambda)(f))(u) = -\varepsilon(a,e)\varepsilon(b,e)f(p(x) \text{ o } p(y)(u) - \varepsilon(a,b)\, p(y) \text{ o } p(x)(u))$.

Using linearity of $f$,

$$= -\varepsilon(a,e)\varepsilon(b,e)f(p(x) \text{ o } p(y)(u)) + \varepsilon(a,e)\varepsilon(b,e)\varepsilon(a,b)f(p(y) \text{ o } p(x)(u)).$$

We now turn to evaluating $(p^*(x)\, p^*(y)(f))(u)$ Applying the formulation of $p^*$

$$(p^*(x) \text{ o } p^*(y)(f))(u) = -\varepsilon(a, b+e)(p^*(y)(f)) \text{ o } (p(x)(u)).$$

Again applying the definition of $p^*(y)$,

$= -\varepsilon(a,b+e)(-\varepsilon(b,e)f(p(y) \text{ o } (p(x)(u))))$.

As a result, $(p^*(x) \text{ o } p^*(y)(f))(u) = \varepsilon(a,b+e)\varepsilon(b,e)f(p(y) \text{ o } p(x)(u))$.

Using the bicharacter property, $\varepsilon(a,b+e) = \varepsilon(a,b)\varepsilon(a,e)$,we arrive at

$$(p^*(x) \text{ o } p^*(y)(f))(u) = \varepsilon(a,b)\varepsilon(a,e)\varepsilon(b,e)f(p(y) \text{ o } p(x)(u)).$$

By an analogous argument,

$$(p^*(y) \text{ o } p^*(x)(f))(u) = \varepsilon(b,a+e)\varepsilon(a,e)f(p(x) \text{ o } p(y)(u)).$$

Using $\varepsilon(b,a+e) = \varepsilon(b,a)\varepsilon(b,e)$, we get

$$(p^*(y) \circ p^*(x)(f))(u) = \varepsilon(b,a)\varepsilon(b,e)\varepsilon(a,e)f(p(x) \circ p(y)(u)).$$

Thus we conclude that,

$(p^*(x) \circ p^*(y) - \varepsilon(a,b)\, p^*(y) \circ p^*(x))(f)(u) = \varepsilon(a,b)\varepsilon(a,e)\varepsilon(b,e)f(p(y) \circ p(x)(u))$

$-\varepsilon(a,b)\varepsilon(b,a)\varepsilon(b,e)\varepsilon(a,e)f(p(x) \circ p(y)(u)).$

Since ,$\varepsilon(a,b)\varepsilon(b,a) = 1$,we arrive at

$= \varepsilon(a,b)\varepsilon(a,e)\varepsilon(b,e)f(p(y) \circ p(x)(u)) - \varepsilon(a,e)\varepsilon(b,e)f(p(x) \circ p(y)(u)).$

Rearranging the terms,

$= -\varepsilon(a,e)\varepsilon(b,e)f(p(x) \circ p(y)(u)) + \varepsilon(a,e)\varepsilon(b,e)\varepsilon(a,b)f(p(y) \circ p(x)(u)).$

This coincides exactly with the expression obtained for $(p^*([x,y]_\lambda)(f))(u)$.

Thus we conclude that,

$$p^*([x,y]_\lambda) = p^*(x) \circ p^*(y) - \varepsilon(a,b)\, p^*(y) \circ p^*(x).$$

As a result, $(V^*, p^*)$

is a $\Gamma$-representation of the $\Gamma$ −Lie color algebra L.

**Corollary 3.9:-** Let L be a $\Gamma$ −Lie color algebra and let $ad : L \to End(L)$be the adjoint $\Gamma$-representation defined by $ad_x(y) = [x,y]_\lambda$,for all homogeneous elements $x \in L_a, y \in L_b, \lambda \in \Gamma$. Consider the linear map $ad^* : L \to End(L^*)$defined by

$$(ad^*(x)(f))(y) = -\varepsilon(a,e)f(ad_x(y)),$$

for all homogeneous element $x \in L_a, y \in L_b, \lambda \in \Gamma, f \in L^*, \deg(f) = e$. Then$(L^*, ad^*)$ is a representation of the $\Gamma$ −Lie color algebra L.

**Proof:-** Let $x \in L_a, y \in L_b, z \in L_c, \lambda \in \Gamma, f \in L^*$.

By definition, $(ad^*(x)(f))(z) = -\varepsilon(a,e)f(ad_x(z))$. Since $ad_x(z) = [x,z]_\lambda$, we arrive at

$$(ad^*(x)(f))(z) = -\varepsilon(a,e)f([x,z]_\lambda).$$

Now compute $(ad^*(x) \circ ad^*(y))(f)(z)$. Applying the formulation of $ad^*$,

$= -\varepsilon(a,b+e)\big(ad^*(y)(f)\big)\big(ad_x(z)\big)$. Again applying the definition of $ad^*$,

$= -\varepsilon(a, b+e)\left(-\varepsilon(b,e)f\left(ad_y\left(ad_x(z)\right)\right)\right)$. As a result,

$(ad^*(x) \text{ o } ad^*(y))(f)(z) = \varepsilon(a, b+e)\varepsilon(b,e)f(ad_y(ad_x(z)))$. Since $ad_x(z) = [x,z]_\lambda$, we arrive at

$= \varepsilon(a, b+e)\varepsilon(b,e)\, f([y,[x,z]_\lambda]_\lambda)$. By an analogous argument,

$$(ad^*(y) \text{ o } ad^*(x))(f)(z) = \varepsilon(b, a+e)\varepsilon(a,e)f([x,[y,z]_\lambda]_\lambda).$$

Thus we conclude that,

$(ad^*(x) \text{ o } ad^*(y) - \varepsilon(a,b)ad^*(y) \text{ o } ad^*(x))(f)(z) = \varepsilon(a, b+e)\varepsilon(b,e)f([y,[x,z]_\lambda]_\lambda)$

$-\varepsilon(a,b)\varepsilon(b,a+e)\varepsilon(a,e)f([x,[y,z]_\lambda]_\lambda)$.

Using the bicharacter properties,

$$\varepsilon(a, b+e) = \varepsilon(a,b)\varepsilon(a,e), \text{and } \varepsilon(b, a+e) = \varepsilon(b,a)\varepsilon(b,e),$$

we arrive at $\varepsilon(a,b)\varepsilon(a,e)\varepsilon(b,e)f([y,[x,z]_\lambda]_\lambda) - \varepsilon(a,b)\varepsilon(b,a)\varepsilon(b,e)\varepsilon(a,e)f([x,[y,z]_\lambda]_\lambda)$.

Since $\varepsilon(a,b)\varepsilon(b,a) = 1$, it follows that

$= \varepsilon(a,e)\varepsilon(b,e)(\varepsilon(a,b)f([y,[x,z]_\lambda]_\lambda) - f([x,[y,z]_\lambda]_\lambda))$.

By the ε-Jacobi identity,

$$[x,[y,z]_\lambda]_\lambda - \varepsilon(a,b)[y,[x,z]_\lambda]_\lambda = [[x,y]_\lambda, z]_\lambda.$$

As a result,

$$\varepsilon(a,b)[y,[x,z]_\lambda]_\lambda - [x,[y,z]_\lambda]_\lambda = -[[x,y]_\lambda, z]_\lambda.$$

Thus we conclude that,

$$(ad^*(x) \text{ o } ad^*(y) - \varepsilon(a,b)ad^*(y) \text{ o } ad^*(x))(f)(z) = -\varepsilon(a,e)\varepsilon(b,e)f([[x,y]_\lambda, z]_\lambda).$$

Since $|[x,y]_\lambda| = a + b$,

we have $\varepsilon([x,y]_\lambda, e) = \varepsilon(a,e)\varepsilon(b,e)$. Thus,

$= -\varepsilon([x,y]_\lambda, e)f(ad_{[x,y]_\lambda}(z))$. As a result,

$= (ad^*([x,y]_\lambda)(f))(z)$. Thus we conclude that,

$$ad^*(x) \circ ad^*(y) - \varepsilon(a,b)ad^*(y) \circ ad^*(x) = ad^*([x,y]_\lambda).$$

As a result, $(L^*, ad^*)$ is a $\Gamma -$ representation of the $\Gamma -$ Lie color algebra L. ■

## 4. Conclusions:

In this paper, we introduced the notion of $\Gamma -$ Lie color algebras and developed several aspects of their representation theory. We presented the fundamental concepts of graded vector spaces, bicharacters, and homogeneous linear maps, which provide the algebraic framework for $\Gamma -$ Lie color algebras. We defined $\Gamma -$representations of $\Gamma -$ Lie color algebras and established basic properties of these $\Gamma -$representations. In particular, we proved that the adjoint action determines a natural representation of a $\Gamma -$ Lie color algebra on itself. Furthermore, we constructed the semidirect product associated with a $\Gamma -$representation and showed that it inherits a $\Gamma -$ Lie color algebra structure. We also investigated dual $\Gamma -$representations and established the dual adjoint $\Gamma -$representation. These constructions extend several classical results from Lie algebras to the setting of $\Gamma -$ Lie color algebras. The results obtained in this work provide a foundation for further studies on derivations, cohomology, extensions, and other structural properties of $\Gamma -$ Lie color algebras.

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