

The ρ_{LM} -Angle in Normed Spaces

Authors Names	ABSTRACT
<p><i>Laith Ahmed Jasim^a</i> <i>Mohammed Yahya Abed^b</i> Publication Date: 30/6/2026 Keywords: Normed Space, Orthogonality, Angle, Norm Derivatives.</p>	<p>Vector angles constitute a foundational element in the analysis of the geometric architecture of linear spaces. Within the framework of inner product spaces, a natural definition of angle emerges through the inner product itself, giving rise to a number of geometrically significant properties. In general normed spaces, however, the absence of an inner product renders the formulation of a suitable notion considerably more complex. Prior research introduced a novel form of orthogonality, designated ρ_{LM}-orthogonality, constructed upon the foundation of ρ_{*}-orthogonality, which itself is derived from norm derivatives. Extending this framework, the present study proposes a novel concept of angle between two nonzero vectors in real normed spaces, termed the ρ_{LM}-angle. The formulation is grounded in the functional ρ_{LM}, combined with an appropriate normalization scheme that accounts for its homogeneity characteristics. A series of fundamental properties pertaining to the ρ_{LM}-angle are subsequently derived and established. Among the results obtained, it is demonstrated that the ρ_{LM}-angle adheres to essential geometric characteristics, namely boundedness, symmetry, and invariance with respect to scalar multiplication. Furthermore, a precise correspondence between orthogonality and the right angle is established, wherein ρ_{LM}-orthogonality is shown to be equivalent to an angle of $\pi/2$. The investigation further examines the connection between the ρ_{LM}-angle and linear dependence. It is established that linearly dependent vectors necessarily yield a zero -angle. Nevertheless, counterexamples drawn from classical spaces, such as l^1, reveal that the converse does not hold in general, thereby exposing a distinctly non-Euclidean geometric behavior inherent to general normed spaces.</p>

1. Introduction

Let $(\tilde{X}, \|\cdot\|)$ be a real vector space with a dimension of at least two. There are several concepts of orthogonality in normed linear space. One of the most well-known notions is Birkhoff orthogonality $s \perp_B r$ if $\|s + tr\| \geq \|s\| \quad \forall t \in \mathbb{R}$ (see [4,5]).

Norm derivatives are one of the possible notions of orthogonality in [2] defined using two functionals, $\rho_{-}, \rho_{+}: \tilde{X} \times \tilde{X} \rightarrow \mathbb{R}$ be defined as follows:

$$\rho_{-}(s, r) = \|s\| \lim_{t \rightarrow 0^{-}} \frac{\|s+tr\| - \|s\|}{t} \quad \text{and} \quad \rho_{+}(s, r) = \|s\| \lim_{t \rightarrow 0^{+}} \frac{\|s+tr\| - \|s\|}{t}$$

The mapping ρ was put forward by Milićić [9], defined according to the formulation presented therein $\rho(s, r) = \langle s, r \rangle_g = \frac{\rho_{-}(s,r) + \rho_{+}(s,r)}{2}$, Nur and Gunawan [11] defining an orthogonality relation grounded in norm derivatives, expressed by the condition presented in their work $\rho_{gg} = \sqrt{|\rho(s, r)| |\rho(r, s)|}$.

^aDepartment of Mathematics, College of Education for Pure Science Kerbala University, Iraq, laith.ahmed@s.uokerbala.edu.iq

^bDepartment of Mathematics, College of Education for Pure Science Kerbala University, Iraq, mohammed.y@uokerbala.edu.iq

In a related contribution, Zamani and Moslehian [13] put forward an orthogonality functional defined by the expression $\wp_\lambda: \check{X} \times \check{X} \rightarrow \mathbb{R}$ by $\wp_\lambda(s, r) = \lambda \wp_-(s, r) + (1 - \lambda) \wp_+(s, r) \forall \lambda \in \mathbb{R}$, under which s is said to be \wp_λ -orthogonality to r , a relation denoted accordingly.

Also Zamani and Moslehian [10] introduced the notion of \wp_* -orthogonality as: $s \perp_{\wp_*} r$ if and only if $\wp_*(s, r) = 0$ where $\wp_*(s, r) = \wp_+(s, r) \wp_-(s, r)$

In a previous study on orthogonality in real normed spaces, we introduced orthogonality using the relation \wp_{LM} (see [1]).

Definition 1.1. [1]: Suppose $(\check{X}, \|\cdot\|)$ represents a real normed vector space then $\forall s, r \in \check{X}$, the functional $\wp_{LM}: \check{X} \times \check{X} \rightarrow \mathbb{R}$ which is defined as $\wp_{LM}(s, r) = \sqrt{|\wp_*(s, r) \wp_*(r, s)|}$, We say that s is \wp_{LM} -orthogonal to r , denoted by $s \perp_{\wp_{LM}} r$ if $\wp_{LM}(s, r) = 0$

Proposition 1.2. [1]: Let $(\check{X}, \|\cdot\|)$ be a real normed vector space for all $s, r \in \check{X}$ and $\alpha, \beta \in \mathbb{R}$. Then

- (a) $\wp_{LM}(s, r) \geq 0$
- (b) $\wp_{LM}(s, 0) = \wp_{LM}(0, r) = 0$
- (c) $\wp_{LM}(s, s) = \|s\|^4$, also $\wp_{LM}(s, s) = \wp_*(s, s)$
- (d) $\wp_{LM}(s, r) = \wp_{LM}(r, s)$
- (e) $\wp_{LM}(\alpha s, \beta r) = |\alpha\beta|^2 \wp_{LM}(s, r)$
- (f) $\wp_{LM}(s, r) \leq \|s\|^2 \|r\|^2$
- (g) $\wp_{LM}(s, ts + r) = \sqrt{[t^2 \|s\|^4 + 2t \|s\|^2 \wp_*(s, r) + \wp_*(s, r)] [\wp_*(ts + r, s)]}$
- (h) $\forall s, r \in \check{X} - \{0\}$, if $s \perp_{\wp_{LM}} r$, then s and r are linearly independent. (linear independence)

The Cauchy-Schwarz inequality gives a natural way to find the angle between two nonzero vectors s and R in a standard inner product space \check{X} . However, in general normed spaces that do not have an inner product, we need to use different methods to describe the geometric relationship between vectors.

In this study, we strengthen the analytical framework by adding the concept of the \wp_{LM} -angle between non-zero vectors. This generalization enables us to measure angles in non-inner product spaces while preserving important geometric properties such as symmetry and invariance under scalar multiplication. The main objectives of this study are to formalize the \wp_{LM} -angle, demonstrate its relationship to \wp_{LM} -perpendicularity, and investigate its relationship to linear dependency. We provide comprehensive illustrative examples in the classical sequence space l^1 to demonstrate remarkable non-Euclidean behavior. In particular, we show that linearly independent vectors can have a zero-angle, indicating that these generalized spaces possess unique parallel-like properties.

Let $(\check{X}, \langle \cdot, \cdot \rangle)$ be an inner product space, the angle $\theta(s, r)$ between two nonzero vectors s and r in \check{X} is given by $\theta(s, r) = \arccos\left(\frac{\langle s, r \rangle}{\|s\| \|r\|}\right)$ where $\|s\| = \langle s, s \rangle^{\frac{1}{2}}$ denotes the induced norm in \check{X} . One may observe that the angle $\theta(s, r)$ in \check{X} then the following basic properties are satisfied (see [8]):

(i) $\theta(s, r) = 0$ if and only if s and r are of the same direction; $\theta(s, r) = \pi$ if and only if s and r are of opposite direction. (Parallelism)

(ii) $\theta(s, r) = \theta(r, s) \forall s, r \in \check{X} - \{0\}$. (Symmetry)

(iii) $\theta(\alpha s, \beta r) = \begin{cases} \theta(s, r), & \alpha, \beta > 0 \\ \pi - \theta(s, r), & \alpha, \beta < 0 \end{cases}$. (Homogeneity)

Nur and Gunawan [12] defining angle between two nonzero vectors s and r belonging to the space \check{X} . This angular measure, designated by the notation $\theta_{gg}(s, r)$, is formally characterized through the following expression: $\theta_{gg}(s, r) = \arccos\left(\frac{\mathfrak{p}_{gg}(s, r)}{\|s\| \|r\|}\right)$. It is worth emphasizing that the resulting angular value is bounded within the closed interval $\left[0, \frac{\pi}{2}\right]$ ensuring that: $0 \leq \theta_{gg}(s, r) \leq \frac{\pi}{2}$

Moreover, it can be readily observed that $\theta_{gg}(s, r)$ satisfies the symmetry property, in the sense that the angle measured from s to r coincides with that measured in the reverse direction.

Zamani and Moslehian [13] define the angle by The number $\theta_{\lambda}(s, r) = \arccos\left(\frac{\mathfrak{p}_{\lambda}(s, r)}{\|s\| \|r\|}\right)$ called λ -angle between the element s, r in a normed linear space. For more information about the topic of angles you can see [3,6,7].

2. The \mathfrak{p}_{LM} -Angle and Its Fundamental Properties

Using the \mathfrak{p}_{LM} -orthogonality established in our previous work, define vectors s and r in $\check{X} - \{0\}$ called \mathfrak{p}_{LM} -angle.

Definition 2.1. Suppose $(\check{X}, \|\cdot\|)$ is a real normed space. The \mathfrak{p}_{LM} -angle between two nonzero vectors s, r in \check{X} is the relation $\theta_{LM}(s, r) = \arccos\left(\frac{\mathfrak{p}_{LM}(s, r)}{\|s\|^2 \|r\|^2}\right)$ where $s, r \neq 0$.

The normalization by $\|s\|^2 \|r\|^2$ is natural since \mathfrak{p}_{LM} is homogeneous of degree two in each variable.

Note: Since $\mathfrak{p}_{LM}(s, r) \geq 0$ by definition, the domain of the arccosine function implies that $0 \leq \theta_{LM}(s, r) \leq \frac{\pi}{2}$. Furthermore, $\theta_{LM}(s, r) = \frac{\pi}{2}$ if and only if $\mathfrak{p}_{LM}(s, r) = 0$, which is exactly equivalent to $s \perp_{\mathfrak{p}_{LM}} r$. Because the angle is invariant under scalar multiplication by negative numbers, $\theta_{LM}(s, r)$ effectively measures the angle between the lines generated by the vectors s and r , rather than the directed vectors themselves.

Theorem 2.2. Suppose $(\ddot{X}, \|\cdot\|)$ is a real normed space and $\forall s, r \in \ddot{X} - \{0\}$ then $s \perp_{\mathfrak{P}_{LM}} r$ if and only if $\theta_{LM}(s, r) = \frac{\pi}{2}$.

Proof: $s \perp_{\mathfrak{P}_{LM}} r \Leftrightarrow \mathfrak{P}_{LM}(s, r) = 0 \Leftrightarrow \frac{\mathfrak{P}_{LM}(s, r)}{\|s\|^2 \|r\|^2} = 0 \Leftrightarrow \arccos\left(\frac{\mathfrak{P}_{LM}(s, r)}{\|s\|^2 \|r\|^2}\right) = \frac{\pi}{2} \Leftrightarrow \theta_{LM}(s, r) = \frac{\pi}{2}$.

Proposition 2.3. Let $(\ddot{X}, \|\cdot\|)$ is a real normed vector space for all s, r in \ddot{X} . Then

(i) $0 \leq \theta_{LM}(s, r) \leq \frac{\pi}{2}$

(ii) $\theta_{LM}(\alpha s, \beta r) = \theta_{LM}(s, r) \forall \alpha, \beta \in \mathbb{R} - \{0\}$

(iii) $\theta_{LM}(s, s) = 0$

Proof

(i) By Proposition 1.1. $\mathfrak{P}_{LM}(s, r) \geq 0$ and $\mathfrak{P}_{LM}(s, r) \leq \|s\|^2 \|r\|^2$ for all $s, r \in \ddot{X}$

We obtain $0 \leq \mathfrak{P}_{LM}(s, r) \leq \|s\|^2 \|r\|^2$

dividing by $\|s\|^2 \|r\|^2$ we get $0 \leq \frac{\mathfrak{P}_{LM}(s, r)}{\|s\|^2 \|r\|^2} \leq 1$

therefore $0 \leq \cos\theta_{LM}(s, r) \leq 1$

hence $0 \leq \theta_{LM}(s, r) \leq \frac{\pi}{2}$

(ii) by define $\theta_{LM}(s, r) = \arccos\left(\frac{\mathfrak{P}_{LM}(s, r)}{\|s\|^2 \|r\|^2}\right)$

Then $\theta_{LM}(\alpha s, \beta r) = \arccos\left(\frac{\mathfrak{P}_{LM}(\alpha s, \beta r)}{\|\alpha s\|^2 \|\beta r\|^2}\right)$

By Proposition 1.2. $\mathfrak{P}_{LM}(\alpha s, \beta r) = |\alpha\beta|^2 \mathfrak{P}_{LM}(s, r)$ for all $s, r \in \ddot{X}$ and all $\alpha, \beta \in \mathbb{R}$

We obtain $\theta_{LM}(\alpha s, \beta r) = \arccos\left(\frac{|\alpha\beta|^2 \mathfrak{P}_{LM}(s, r)}{|\alpha|^2 |\beta|^2 \|s\|^2 \|r\|^2}\right) = \arccos\left(\frac{\mathfrak{P}_{LM}(s, r)}{\|s\|^2 \|r\|^2}\right) = \theta_{LM}(s, r)$

hence $\theta_{LM}(\alpha s, \beta r) = \theta_{LM}(s, r)$

(iii) $\mathfrak{P}_{LM}(s, s) = \|s\|^4$

$\theta_{LM}(s, s) = \arccos\left(\frac{\mathfrak{P}_{LM}(s, s)}{\|s\|^2 \|s\|^2}\right) = \arccos\left(\frac{\|s\|^4}{\|s\|^4}\right) = \arccos(1) = 0$

Theorem 2.4. Let $(\ddot{X}, \|\cdot\|)$ be a real normed vector space. Then $\theta_{LM}(s, r) = \theta_{LM}(r, s)$ for all s, r in \ddot{X}

Proof: By Proposition 1.2. $\mathfrak{P}_{LM}(s, r) = \mathfrak{P}_{LM}(r, s)$ for all $s, r \in \ddot{X}$,

Dividing by $\|s\|^2 \|r\|^2$ we get $\frac{\mathfrak{P}_{LM}(s, r)}{\|s\|^2 \|r\|^2} = \frac{\mathfrak{P}_{LM}(r, s)}{\|r\|^2 \|s\|^2}$

therefore $\arccos\left(\frac{\mathfrak{P}_{LM}(s,r)}{\|s\|^2\|r\|^2}\right) = \arccos\left(\frac{\mathfrak{P}_{LM}(r,s)}{\|r\|^2\|s\|^2}\right)$ which yields $\theta_{LM}(s, r) = \theta_{LM}(r, s)$

Theorem 2.5. Suppose $(\ddot{X}, \|\cdot\|)$ is a real normed vector space and s, r be linearly dependent, then $\theta_{LM}(s, r) = 0$

Proof: let $r = \alpha s$, $\forall s, R$ in $\ddot{X} - \{0\}$ and α in $\dot{R} - \{0\}$.

We have $\theta_{LM}(s, r) = \arccos\left(\frac{\mathfrak{P}_{LM}(s,r)}{\|s\|^2\|r\|^2}\right) = \arccos\left(\frac{\mathfrak{P}_{LM}(s,\alpha s)}{\|s\|^2\|\alpha s\|^2}\right)$

By Proposition 1.2. $\mathfrak{P}_{LM}(\alpha s, \beta r) = |\alpha\beta|^2 \mathfrak{P}_{LM}(s, r) \forall s, r$ in \ddot{X} and α, β in \dot{R}

Thus $\theta_{LM}(s, r) = \arccos\left(\frac{|\alpha^2| \mathfrak{P}_{LM}(s,s)}{\|s\|^2 |\alpha^2| \|s\|^2}\right) = \arccos\left(\frac{\mathfrak{P}_{LM}(s,s)}{\|s\|^2 \|s\|^2}\right) = \arccos(1) = 0$

Remark 2.6. (Non-Euclidean Behavior of the \mathfrak{P}_{LM} -Angle). In Euclidean spaces (inner product spaces), an angle of zero between two vectors strictly implies that they are linearly dependent. However, in general normed spaces, the converse of Theorem 2.5 is not necessarily true. The \mathfrak{P}_{LM} -angle successfully detects parallel-type behavior in the geometric structure of the space itself, meaning that linearly independent vectors can exhibit an angle of zero. This highlights the rich, non-Euclidean nature of orthogonality in spaces like l_1 .

Example 2.7. let $s = (2, 1, 0)$ and $r = (1, 2, 0)$ if $\|(s_1, s_2, s_3)\| = |s_1| + |s_2| + |s_3|$ in the space $\ddot{X} = l^1$, then $\|s + tr\| = \|2 + t, 1 + 2t\| = |2 + t| + |1 + 2t| = |3 + 3t|$

and $\|r + ts\| = \|1 + 2t, 2 + t\| = |1 + 2t| + |1 + 2t| = |3 + 3t|$

$$\mathfrak{P}_+(s, r) = \|s\| \lim_{t \rightarrow 0^+} \frac{\|s+tr\| - \|s\|}{t} = (3) \lim_{t \rightarrow 0^+} \frac{|3+3t|-3}{t} = 3(3) = 9$$

$$\mathfrak{P}_-(s, r) = \|s\| \lim_{t \rightarrow 0^-} \frac{\|s+tr\| - \|s\|}{t} = (3) \lim_{t \rightarrow 0^-} \frac{|3+3t|-3}{t} = 3(3) = 9$$

$$\mathfrak{P}_+(r, s) = \|r\| \lim_{t \rightarrow 0^+} \frac{\|r+ts\| - \|r\|}{t} = (3) \lim_{t \rightarrow 0^+} \frac{|3+3t|-3}{t} = 3(3) = 9$$

$$\mathfrak{P}_-(r, s) = \|r\| \lim_{t \rightarrow 0^-} \frac{\|r+ts\| - \|r\|}{t} = (3) \lim_{t \rightarrow 0^-} \frac{|3+3t|-3}{t} = 3(3) = 9$$

Therefore $\mathfrak{P}_*(s, r) = \mathfrak{P}_+(s, r) \mathfrak{P}_-(s, r) = (9)(9) = 81$

and $\mathfrak{P}_*(r, s) = \mathfrak{P}_+(r, s) \mathfrak{P}_-(r, s) = (9)(9) = 81$

Thus $\mathfrak{P}_{LM}(s, r) = \sqrt{|\mathfrak{P}_*(s, r) \mathfrak{P}_*(r, s)|} = \sqrt{|(81)(81)|} = 81$

$$\theta_{LM}(s, r) = \arccos\left(\frac{\mathfrak{P}_{LM}(s,r)}{\|s\|^2\|r\|^2}\right) = \arccos\left(\frac{81}{(9)(9)}\right) = \arccos(1) = 0$$

yields $\theta_{LM}(s, r) = 0$ but s, r are clearly linearly independent

Example 2.8. Let $\ddot{X} = \dot{R}^2$ is a real normed space with the norm $\|s\| = |s_1| + |s_2|$ where $s = (1, 0)$ and $r = (0, 1)$

we have $\mathfrak{p}_+(s, r) = \|s\| \lim_{t \rightarrow 0^+} \frac{\|s+tr\| - \|s\|}{t}$ and $\mathfrak{p}_-(s, r) = \|s\| \lim_{t \rightarrow 0^-} \frac{\|s+tr\| - \|s\|}{t}$

Thus $\mathfrak{p}_+(s, r) = (1) \lim_{t \rightarrow 0^+} \frac{1+t-1}{t} = 1$ and $\mathfrak{p}_-(s, r) = (1) \lim_{t \rightarrow 0^-} \frac{1-t-1}{t} = -1$

$\mathfrak{p}_+(r, s) = (1) \lim_{t \rightarrow 0^+} \frac{t+1-1}{t} = 1$ and $\mathfrak{p}_-(r, s) = (1) \lim_{t \rightarrow 0^-} \frac{-t+1-1}{t} = -1$

$$\mathfrak{p}_{LM}(s, r) = \sqrt{|\mathfrak{p}_*(s, r) \mathfrak{p}_*(r, s)|} = \sqrt{|(1)(-1)(1)(-1)|} = \sqrt{|1|} = 1$$

$$\theta_{LM}(s, r) = \arccos\left(\frac{\mathfrak{p}_{LM}(s, r)}{\|s\| \|r\|}\right) = \arccos\left(\frac{1}{(1)(1)}\right) = \arccos(1) = 0$$

Although s, r are linearly independent, the \mathfrak{p}_{LM} -angle detects parallel-type behavior, showing that orthogonality in l^1 is essentially non-Euclidean.

References

- [1] Ahmed Jasim, L. A., & Abed, M. Y. (2025). New kind of orthogonality in normed spaces by using \mathfrak{p}_{LM} -orthogonality. *F1000Research*, 14, 1486.
- [2] Amir, D. (1986). *Characterizations of inner product spaces* (Vol. 20, p. 3). Basel: Birkhäuser.
- [3] Balestro, V., Horváth, Á. G., Martini, H., & Teixeira, R. (2017). Angles in normed spaces. *Aequationes mathematicae*, 91(2), 201-236.
- [4] Birkhoff, G. (1935). Orthogonality in linear metric spaces. *Duke Mathematical Journal*, 1(2), 169–172.
- [5] Diminnie, C. R. (1983). A new orthogonality relation for normed linear spaces. *Mathematische Nachrichten*, 114(1), 197-203.
- [6] Gunawan, H., Jamaludin, M., & Pratamadirdja, M. D. (2021). On Birkhoff angles in normed spaces. *arXiv preprint arXiv:2107.09454*.
- [7] Gunawan, H., Lindiarni, J., & Neswan, O. (2008). P-, I-, g-, and D-angles in normed spaces. *J. Math. Fund. Sci*, 40(1), 24-32.
- [8] Milicic, P. M. (2007). On the B-angle and g-angle in normed spaces. *J. Inequal. Pure Appl. Math*, 8(3), 1-9.
- [9] Miličić, P. M. (1987). Sur la G-orthogonalité dans les espaces normés. *Matematički Vesnik*, 39, 325-334.
- [10] Moslehian, M. S., Zamani, A., & Dehghani, M. (2017). Characterizations of smooth spaces by \mathfrak{p}_* -orthogonality. *arXiv preprint arXiv:1705.07032*.
- [11] Nur, M., & Gunawan, H. (2019). A new orthogonality and angle in a normed space. *Aequationes mathematicae*, 93(3), 547-555.
- [12] Nur, M., Gunawan, H., & Neswan, O. (2018). A formula for the g-angle between two subspaces of a normed space. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 59(1), 133-143.
- [13] Zamani, A., & Moslehian, M. S. (2019). An extension of orthogonality relations based on norm derivatives. *The Quarterly Journal of Mathematics*, 70(2), 379-393.