

Some Results and Properties of Fuzzy Non-Adjacency Compatible Vertices Topology on Fuzzy Digraphs

| <i>Authors Names</i> | ABSTRACT |
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| <p><i>Iqbal Salman Hussein^a</i> <i>Asmhan Fliih Hassan^b</i></p> <p><i>Publication data: 30 / 6 /2026</i></p> <p><i>Keywords: Fuzzy digraph, Fuzzy Non-Adjacency Compatible Vertices Topology , Alexandroff space.</i></p> | <p>This paper introduces some results and properties of fuzzy non-adjacency compatible vertices topology related to fuzzy directed graphs, we have studied and analyzed a number of associated properties and also discussed them in the context of some important and specific types of fuzzy digraph labels. Our aim is to give basic steps towards investigating some properties of fuzzy directed graphs through this our new fuzzy topology. Through rigorous investigation and the establishment of foundational propositions, corollaries, and some examples, this research clarified several important topological characteristics, such as relations of fuzzy separation axioms, fuzzy compactness, Alexandroff space properties, and conditions for discreteness.</p> |

1. Introduction

Fuzzy graph is one of the useful mathematical tools that enable users to represent fuzzy relationships between different elements in a flexible and precise manner [3,6]. The nature of fuzziness is suitable for any environment, making it ideal in situations where it is difficult to define relationships in a traditional manner, thus facilitating problem prediction and resolution. Since it provides a mathematical framework that allows dealing with complex phenomena in a more flexible manner compared to classical models that rely on specific relationships. Fuzzy graphs can be used in many fields, including engineering and social sciences, due to their ability to estimate changing and uncertain relationships. Sources [1,4,12] provide examples of topological spaces within a graph theory or fuzzy topological spaces.

The concept of fuzzy topological spaces was introduced in 1968 by C.L. Chang [7]. In 1975, Atef and his colleagues [8] presented a novel type of fuzzy topological notions and structures using fuzzy graph and named them "fuzzy topological graphs", where fuzzy graphs are studied on a specific class of fuzzy sets. Due to their common applications in many fields, it becomes necessary to study the relationship between fuzzy topological spaces and fuzzy graphs.

In 2022, A.M. Alzubaidi and M. Dammak [2] introduced graphic topology on fuzzy graphs. Subsequently, in 2023, P. S. Gholap and V. E. Nikumbh [9] developed fuzzy vertex and fuzzy edge topologies on simple and connected fuzzy graph, utilizing the adjacency relations, also they defined fuzzy neighborhood topology on simple fuzzy graphs.

Building on this, in 2024, researchers in [10,11] defined and investigated the interrelationships of various fuzzy topological spaces on fuzzy digraphs. they introduced left (right) fuzzy vertex and edge topologies, based on adjacency relations of fuzzy vertices and edges in fuzzy digraphs.

Although many researchers have studied fuzzy topological spaces on different fuzzy graphs, there is not enough literature on this concept and it is still limited. The use of fuzzy topological structures has

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allowed us to study multiple aspects of fuzzy digraphs, which is the motivation for introducing our new fuzzy topology on some fuzzy digraphs.

The primary aim of this our study is to delineate fuzzy topology within the framework of the fuzzy digraphs. In our recent research, we introduce properties and results designate in Fuzzy Non-Adjacency Compatible Vertex Topology. We shall examine the benefits of Fuzzy Non-Adjacency Compatible Vertex Topology concerning essential categories of main types of fuzzy directed graphs. This our new paper is organized as the following: in Part 2, we displayed main definitions of fuzzy topology, fuzzy graph and Fuzzy Non-Adjacency Compatible Vertex Topology. In Part 3, we discussed some properties and results through rigorous investigation and the establishment of foundational propositions, corollaries, and some examples finally the conclusions.

2. Preliminaries

This section introduces fuzzy directed graph theory and fuzzy topology basic definitions and preliminaries. All of fuzzy concepts are widely used and are found in references like [3,5,6,7,11].

Definition 2.1 ([11]). For a universe S , any map $\gamma: S \rightarrow [0, 1]$ is regarded as a fuzzy sub-set of the set S , while every subset of $S \times S$ is named as fuzzy relation on S . If sets D and G are fuzzy on the same universe, their join and meet are defined point-wise as: $(D \vee G)(x) = \max\{D(x), G(x)\}$, $(D \wedge G)(x) = \min\{D(x), G(x)\} \forall x \in S$

Definition 2.2.([7]) suppose the set X is a non-empty fuzzy. Then a fuzzy topology on it is the collection \mathcal{T}_F of sub-sets of X ($\mathcal{T}_F \subseteq P(X)$) that satisfies the following three conditions:

- (i) $X \in \mathcal{T}_F$
- (ii) For any $A, B \in \mathcal{T}_F$, their intersection $A \cap B \in \mathcal{T}_F$
- (iii) For any family $\{A_i\}_{i \in I}$ of elements in \mathcal{T}_F , their union $\bigcup_{i \in I} A_i \in \mathcal{T}_F$.

Any member A of \mathcal{T}_F is called a fuzzy-open-set. When $\mathcal{T}_F = P(X)$ then it is known as discrete fuzzy topology, and when $\mathcal{T}_F = \{\emptyset, X\}$ is known as trivial fuzzy topology.

Definition 2.3 ([3]) The fuzzy directed graph $\mathfrak{D}_f = (\mathcal{V}, \gamma, w)$ is formed by a vertex set \mathcal{V} with a vertex-membership map $\gamma: \mathcal{V} \rightarrow [0, 1]$ and a symmetric edge-membership map $w: \mathcal{V} \times \mathcal{V} \rightarrow [0, 1]$ that obey the constraint $w(a, b) \leq \gamma(a) \wedge \gamma(b)$ for all $a, b \in \mathcal{V}$; the value $\gamma(a)$ represents the degree to which a belongs to the graph, whereas $w(a, b)$ measures the strength of the edge joining a and b . $H = (U, \rho, \nu)$ with $U \subseteq V$ is called a partial fuzzy subgraph when $\rho \leq \gamma$ and $\nu \leq w$, and it becomes spanning if $U = V$. A path of length n is a sequence of distinct vertices $a_0, a_1, a_2, \dots, a_n$ satisfying $w(a_{i-1}, a_i) > 0$ for every $1 \leq i \leq n$; two vertices are regarded as connected whenever such a path exists, and the graph is named as linked if this property holds for every vertices pair.

The strength of given path between the vertices l and d is defined as: $\bigwedge_{i=1}^n w(l_{i-1}, l_i)$. Moreover, the strength of connectedness between the vertices l and d is determined via the maximum strength of total possible paths connecting them, symbolized by $w^\infty(l, d)$. Then $\mathfrak{D}_f = (\mathcal{V}, \gamma, w)$ is classified as connected if $w^\infty(l, d) > 0$ for all pair of vertices $l, d \in \mathcal{V}$.

Definition 2.4 ([6]). Fuzzy directed graph $\mathfrak{D}_f = (\mathcal{V}, \gamma, w)$ is complete when every potential edge attains its maximal admissible weight, namely $\varpi(l, d) = \gamma(l) \wedge \gamma(d) \forall l, d \in \mathcal{V}$.

Definition 2.5 ([6]). Fuzzy directed graph $\mathfrak{D}_f = (\mathcal{V}, \gamma, w)$ is complete-bipartite if the vertex fuzzy set can be partitioned to two non-empty, disjoint subsets V_1 and V_2 such that $w(l, d) = 0$ whenever l, d lie in same subset, and $w(l, d) = \gamma(l) \wedge \gamma(d)$, whenever $l \in \mathcal{V}_1$ and $d \in \mathcal{V}_2$.

Definition 2.6 ([6]). Consider the structure (X, \mathcal{T}_f) , know as a fuzzy topological space. Suppose that β is a sub-family of \mathcal{T}_f , then β is termed as base of \mathcal{T}_f , if any element of \mathcal{T}_f can represented union of elements selected from β . Similarly, a sub-family S within \mathcal{T}_f is termed a sub-base if the collection formed by finite-intersections among its elements constitutes a base of \mathcal{T}_f . Furthermore, for two given fuzzy topologies \mathcal{T}_{f_1} and \mathcal{T}_{f_2} on a set X , we describe \mathcal{T}_{f_1} as being coarser than \mathcal{T}_{f_2} if it satisfies the inclusion $\mathcal{T}_{f_1} \subseteq \mathcal{T}_{f_2}$ and conversely, \mathcal{T}_{f_2} as finer than the fuzzy topology \mathcal{T}_{f_1} .

Definition 2.7 ([6]) Fuzzy topology \mathcal{T}_f can be named generated by a sub-family S of the fuzzy sub-sets within X , provided each element of \mathcal{T}_f can be represented as union of finite -intersections of elements from the family S .

Definition 2.8. ([6]) Consider the fuzzy directed graph $\mathfrak{D}_f = (\mathcal{V}, \gamma, w)$. We say that \mathfrak{D}_f is a fuzzy bipartite graph if the vertex fuzzy set \mathcal{V} can be partitioned to exactly two non-empty, disjoint subsets V_1 and V_2 , meaning $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, and the membership value of any edge connecting vertices from the same subset is equal to zero. Formally, this condition implies that for each vertex $a \in \mathcal{V}_1$ and each vertex $b \in \mathcal{V}_2$, we have the edge membership function $\mu(a, b) \neq 0$, while it is zero otherwise.

Definition 2.9. ([6]) A fuzzy directed graph without loops or parallel fuzzy edges will be called a simple fuzzy directed graph. By a loop we mean an edge incident to a vertex that it repeats on itself. Furthermore, also parallel fuzzy edges are formed when at least two distinct edges connect the same two distinct vertices.

Definition 2.10.([6]) A fuzzy path P_n is defined as a fuzzy directed graph with exactly n vertices and exactly $n - 1$ edges. Its vertices can be denoted sequentially by v, v_2, \dots, v_n , while the edges of the fuzzy path connect consecutive vertices v_r and v_{r+1} for $r = 1, 2, 3, \dots, n - 1$. Furthermore, a fuzzy path is said to be closed when $v_0 = v_n$. A fuzzy trial, defined similarly, is called closed if it contains at least one edge that forms a cycle.

Definition 2.11.([6]) Fuzzy directed graph \mathfrak{D}_f is termed fuzzy forest if it has spanning fuzzy sub-graph $F = (\gamma, \nu)$ forming a forest structure, wherein all arcs (x, y) with $w(x, y) < \nu^\infty(x, y)$ do not belong to F . Stated differently, for any two vertices x, y in G connected by an edge not included in F , must be a path connecting x and y whose strength is greater than $\mu(x, y)$. Additionally, if a fuzzy forest is connected, we then specifically call it a fuzzy tree.

Definition 2.12: ([5]) : Let $\mathfrak{D}_f = (V, \gamma, w)$ be any fuzzy digraph, on a vertices fuzzy set $V(\mathfrak{D}_f)$, then we can define a Fuzzy Non-Adjacency Compatible Vertices Topology (\mathcal{T}_{2f}) and brought on by sub-basis \mathcal{S}_{2f} whose components are the sets $H \subseteq V(\mathfrak{D}_f), |H| \leq 2$; if $(a, \gamma(a)) \in W$ and the vertex $(b, \gamma(b))$ is non-adjacent with vertex $(a, \gamma(a))$, such that $(a, \gamma(a))$ and $(b, \gamma(b))$ connected by a fuzzy path of length two in the same direction then $(b, \gamma(b)) \in W$. i.e. via a relation P which is $P[a] = \{(a, \gamma(a))\}$ or $\{(a, \gamma(a)), (b, \gamma(b))\}$, where $\varpi(a, b) = 0 \wedge$ construct a fuzzy path of length two in same direction}, Then the set $\mathcal{S}_{2f} = \{P[a]: P[a] \subseteq V(\mathfrak{D}_f)\}$ constructs a sub-base to a fuzzy topology on $V(\mathfrak{D}_f)$. And let \mathcal{B}_{2f} represents the finite-intersection of elements of sub-base \mathcal{S}_{2f} . Clearly, \mathcal{B}_{2f} constructs a base.

Then collection \mathcal{T}_f of all union of elements of \mathcal{B}_f represents a fuzzy topology on fuzzy set $V(\mathcal{D}_f)$, which named as a Fuzzy Non-Adjacency Compatible Vertices Topology associated by fuzzy digraph \mathcal{D}_f , the pair $(V(\mathcal{D}_f), \mathcal{T}_f)$ is a Non-Adjacency Compatible Vertices Topological Space associated by fuzzy digraph \mathcal{D}_f . The elements of \mathcal{T}_f are named as \mathcal{T}_f -open fuzzy sets and the complement are named as \mathcal{T}_f -closed-fuzzy set.

3. Main Results and Properties of Fuzzy Non Adjacency Compatible Vertices Topology on Fuzzy Digraphs.

Proposition 3.1: If $\mathcal{D}_f=(V,\gamma,\omega)$ is a fuzzy directed graph, then the pair (V, \mathcal{T}_{2f}) forms an Alexandroff space.

Proof: To show this, it suffices to prove that the intersection of arbitrary collection of elements from \mathcal{S}_{2f} is open. Consider any set $S \subseteq V$. Suppose $v \in \bigcap_{u \in S} P[u]$. This implies $v \in P[u]$ for each $u \in S$. Since both $P[v]$ and S are finite sets. if S is finite, then the intersection is simply the intersection of finite collection of open sets, thus open itself.

Now, for each vertex $(v \in V)$ is a fuzzy directed graph $\mathcal{D}_f=(V,\gamma,\omega)$, consider the intersection of all open sets that contain v . This intersection represents the smallest open set containing the vertex v . We will call it \dot{D}_v and the collection $\mathcal{B}_{2f} = \{\dot{D}_v | v \in V\}$ is minimal base for the topological space (V, \mathcal{T}_{2f}) .

Proposition 3.2: Let $\mathcal{D}_f = (\mathcal{V}, \gamma, w)$ be a fuzzy directed graph, then we have $\dot{D}_v = \bigcap_{u \in R[v]} P[u]$ and so \dot{D}_v is finite for every $v \in \mathcal{V}$.

Proof: Since \dot{D}_v is smallest open set containing v and \mathcal{S}_{2f} is a sub-basis if \mathcal{T}_{2f} we have $\dot{D}_v = \bigcap_{w \in S} P[w]$ for some subset S of \mathcal{V} . this implies that $v \in P[w]$ for each $w \in S$. therefore $\subseteq P[v]$ and so $v \in \bigcap_{w \in R[v]} P[w]$, then there exists a bi248 \dot{D}_v

Now by definition of \dot{D}_v , the proof is complete.

Corollary 3.3: Let $\mathcal{D}_f = (\mathcal{V}, \gamma, \omega)$ be a fuzzy directed graph. then for every $v, w \in \mathcal{V}$ we have $w \in \dot{D}_v$ if and only if $P[v] \subseteq P[w]$. Equivalently $\dot{D}_v = \{w \in V | P[v] \subseteq P[w]\}$.

Proof: By the proposition above $w \in \dot{D}_v$ if and only if $w \in P[u]$ for each $u \in P[v]$ if and only if $u \in P[w]$ for each $u \in P[v]$

Remark 3.4: Suppose that $\mathcal{D}_f=(V, y, \omega)$ is a fuzzy directed graph, then (V, \mathcal{T}_{2f}) is a discrete topology space if and only if $P[v] \supseteq P[u]$ and $P[u] \supseteq P[v]$ for every distinct pair of vertices $v, u \in V$.

Remark 3.5: We also know that an Alexandroff topological space is \mathcal{T}_1 if and only if it is discrete, this mean that if and only if $P[v] \neq P[u]$ for each unique pair of vertices $u, v \in V$, then the graph $\mathcal{D}_f=(V, y, w)$ has \mathcal{T}_0 independent topology (V, \mathcal{T}_{2f}) . Assume we have a tree $\mathcal{T}=(V, y, w)$ if and only if $P[v] \neq P[u]$ for each $v, u \in V$ such that $v \neq u$ and $\deg v = \deg u = 1$ then (V, \mathcal{T}_{2f}) is a \mathcal{T}_0 - space.

Remark 3.6: It is easy to check if $\mathcal{D}_{f1} = (V_1, y_1, \omega_1)$ and $\mathcal{D}_{f2} = (V_2, y_2, \omega_2)$ are isomorphic directed graphs, then topological space (V_1, \mathcal{T}_{2f1}) and (V_2, \mathcal{T}_{2f2}) are homeomorphic. The converse is not true, in general, for example: C_n when $n \geq 4$ and the simple directed graph has $n \geq 2$ isolated vertex, are

not isomorphic graphs, but their corresponding independent topologies are both discrete and hence homeomorphic.

Proposition 3.7: Consider a fuzzy directed graph $\mathfrak{D}_f = (V, \gamma, \omega)$ the space (V, \mathcal{T}_{2f}) is a compact independent topological space if and only if V is finite.

Proof: From proposition 3.2 each set \mathring{D}_v is finite for all vertices $v \in V$.

Definition 3.8: In a fuzzy directed graph \mathfrak{D}_f , let $F \subseteq V(\mathfrak{D}_f)$, then notation $\mathfrak{D}_f - F$ denotes the sub graph created by removing the vertices in set F . A cut-vertex of \mathfrak{D}_f is a vertex whose removal increases the number of components of the graph.

Specifically, a vertex $v \in V(\mathfrak{D}_f)$ is a cut-vertex if the sub graph $\mathfrak{D}_f - \{v\}$ has more components than the original \mathfrak{D}_f . A vertex cut in a connected fuzzy directed graph \mathfrak{D}_f is a set $H \subseteq V(\mathfrak{D}_f)$ whose removal results in the graph having more than one component. A vertex cut H is minimal if none of its proper subsets can function as vertex cut.

Proposition 3.9: If v be a cut vertex in a fuzzy directed graph $\mathfrak{D}_f = (V, \gamma, \omega)$, which does not necessarily need to be connected, then the singleton set $\{v\}$ must be an element of the fuzzy topology \mathcal{T}_{2f} .

Proof: Let $v \in V(\mathfrak{D}_f)$ such that v is a cut vertex, then $\{v\}$ must be an element of the fuzzy topology \mathcal{T}_{2f} , because v belong to more than one set of sub-base, then the intersection of the sets in base will appeared the singleton set $\{v\}$.

Proposition 3.10: let $\mathfrak{D}_f = (V, \gamma, \omega)$ be a connected fuzzy directed graph and M is minimal vertex cut in \mathfrak{D}_f then $M \in \mathcal{T}_{2f}$.

Proof: Suppose that $\mathfrak{D}_f - M$ has $k \geq 2$ components, say: $\mathfrak{D}_{f_i} = (V_i, \gamma_i, \omega_i)$ for $i = 1, 2, 3, \dots, k$. Every vertex $v \in V$ must be adjacent to vertices of at least two different components, say \mathfrak{D}_{f_1} and \mathfrak{D}_{f_2} , because M is a minimal vertex cut. Suppose that $\{u_1, \dots, u_{k_1}\} = P[v] \cap V_1$ and $\{w_1, \dots, w_{k_2}\} = P[v] \cap V_2$ then we have $v \in \bigcap_{i=1}^{k_1} P[u_i] \subseteq M \cup V_1$ and $v \in \bigcap_{i=1}^{k_2} P[w_i] \subseteq M \cup V_2$ and so $v \in (\bigcap_{i=1}^{k_1} P[u_i]) \cap (\bigcap_{i=1}^{k_2} P[w_i]) \subseteq M \cup (V_1 \cap V_2) = M$ that is v is an interior point of M .

Theorem 3.11: If H is a simple connected spanning fuzzy directed sub graph of fuzzy directed graph \mathfrak{D}_f , then the fuzzy Non-Adjacency Vertex topology generated by fuzzy directed graph H is finer than the Fuzzy Non-Adjacency vertex topology generated by fuzzy directed graph \mathfrak{D}_f .

Theorem 3.12: If two fuzzy directed graph $\mathfrak{D}_{f_1} = (V_1, \gamma_1, \omega_1)$ and $\mathfrak{D}_{f_2} = (V_2, \gamma_2, \omega_2)$ are isomorphic, then the Fuzzy Non-Adjacency Vertex topology \mathcal{T}_{2f_1} and \mathcal{T}_{2f_2} generated by \mathfrak{D}_{f_1} and \mathfrak{D}_{f_2} respectively are homeomorphic.

Proof: Since $\mathfrak{D}_{f_1} = (V_1, \gamma_1, \omega_1)$ and $\mathfrak{D}_{f_2} = (V_2, \gamma_2, \omega_2)$ are isomorphic, then there exists a bijective mapping

$f: \mathfrak{D}_{f_1} \rightarrow \mathfrak{D}_{f_2}$ such that $\gamma_1(a) = \gamma_2(f(a)) \forall a \in V_1$ and $\omega_1(a, b) = \omega_2(f(a), f(b)) \forall a, b \in V_1$ thus $S_1 = \{P[e]: (e, \omega_1(e)) \in E_1\}$ and $S_2 = \{P[f(e)]: ((f(e), \omega_2(f(e))) \in E_2\}$ are equivalent. so the Fuzzy Non-Adjacency Vertex topologies \mathcal{T}_{2f_1} and \mathcal{T}_{2f_2} generated by sub-basis S_1 and S_2 respectively are homeomorphic.

Remark 3.13: If two fuzzy directed graph $\mathfrak{D}_{f_1} = (V_1, \gamma_1, \omega_1)$ and $\mathfrak{D}_{f_2} = (V_2, \gamma_2, \omega_2)$ are weak isomorphic, then the Fuzzy Non-Adjacency Vertex topology \mathbb{T}_{2f_1} generated by \mathfrak{D}_{f_1} and the Fuzzy Non-Adjacency Vertex topology \mathbb{T}_{2f_2} generated by \mathfrak{D}_{f_2} may not be homeomorphic.

Proposition 3.14: Let \mathfrak{D}_f be a fuzzy directed graph and $x \in V$, then $U_x \cap N_x = \emptyset$ for all $y \in N_x$.

Proof: If $y \in U_x \setminus \{x\}$, then $N_x \subseteq N_y$ and $y \neq x$. Since $N_x \neq \emptyset$ there exist $z \in N_x$ and so $z \in N_y$, therefore, $d(x, y) = 2$ and we get: $U_x \subseteq \{x\} \cup \{y \in V; d(x, y) = 2\}$, then $U_x \cap N_x = \emptyset$ and also $N_x \subseteq U_x^c$. If $y \in N_x$, then $U_y \subseteq N_x$. so, $U_y \subseteq U_x^c$, hence $U_x \cap U_y = \emptyset$.

Proposition 3.15: Let \mathfrak{D}_f be a fuzzy directed graph and $x \in V^*$, then $y \in \{x\}$ if and only if $N_y \subseteq N_x$ this means $\overline{\{x\}} = \{y \in V^*; N_y \subseteq N_x\}$.

Proof: $y \in \overline{\{x\}}$ if and only if for all open set Z containing y , $Z \cap \{x\} \neq \emptyset$, but this is equivalent to $U_x \cap \{x\} \neq \emptyset$; so $y \in \overline{\{x\}}$ if and only if $x \in U_y$ and the

Proposition 3.16: Let $\mathfrak{D}_f = (V, \gamma, \omega)$ be fuzzy directed graph, then $U = \{x \in V^*; \deg(x) = \Delta\}$ is an open set for (V^*, \mathbb{T}_{2f}) .

Proof: Suppose that $x \in U$, we will prove that $x \in U_x \subseteq U$. Let $y \in U_x$ from corollary 3.15 $\deg(x) \leq \deg(y)$ and so $\deg(y) = \Delta$, hence $y \in U$ that is $U_x \subseteq U$ and so $x \in U_x \subseteq U$.

Proposition 3.17: Suppose that $\mathfrak{D}_f = (V, \gamma, \omega)$ is a fuzzy directed graph, then $V = \{a \in V^*, \deg(a) = \beta\}$ is a closed set for (V^*, \mathbb{T}_{2f}) .

Proof: We will prove that $v \in V^*$. Since (V^*, \mathbb{T}_{2f}) is an Alexanderoff space, we have any union of closed sets is a closed set and so $\bar{V} = \bigcup_{a \in V} \overline{\{a\}}$. Let $y \in \bar{V}$ there exists $a \in V$ such that $y \in \overline{\{a\}}$, from proposition 3.15, $N_y \subseteq N_a$, then $\deg(y) \leq \deg(a)$ and so, $\deg(y) = \beta$, hence $y \in V$ and the proof follows.

Proposition 3.18: Let $\mathfrak{D}_f = (V, \gamma, \omega)$ be connected directed graph and C is a minimal vertex cut in \mathfrak{D}_f , then $C \in \mathbb{T}_{2f}$.

Proof: Suppose that $\mathfrak{D}_f - C$ has $k \geq 2$ components, say $\mathfrak{D}_{f_i} = (V_i, \gamma_i, \omega_i)$ for $i = 1, 2, 3, \dots, k$. Every vertex $x \in C$ must be adjacent to vertices of at least two different components, say \mathfrak{D}_{f_1} and \mathfrak{D}_{f_2} , because C is a minimal vertex cut. Suppose that $\{y_1, y_2, \dots, y_{k_1}\} = B_x \cap V_1$ and $\{z_1, z_2, \dots, z_{k_2}\} = B_x \cap V_2$, then we have $x \in \bigcap_{i=1}^{k_1} B_{y_i} \subseteq C \cup V_1$ and $x \in \bigcap_{i=1}^{k_2} B_{z_i} \subseteq C \cup V_2$, and so:

$$x \in \left(\bigcap_{i=1}^{k_1} B_{y_i}\right) \cap \left(\bigcap_{i=1}^{k_2} B_{z_i}\right) \subseteq C \cup (V_1 \cap V_2) = C, \text{ that is } x \text{ is an interior point of } C.$$

Proposition 3.19: Let $\mathfrak{D}_f = (V, \gamma, \omega)$ be a finite graph and $\mathbb{T}_{2f}^c = \{U^c \mid U \in \mathbb{T}_{2f}\}$. If $\overline{\mathfrak{D}_f}$ is a connected graph and $\mathbb{T}_{\overline{\mathfrak{D}_f}} = \mathbb{T}_{\mathfrak{D}_f}^c$, then (V, \mathbb{T}_{2f}) is a discrete topological space.

Proof : Let Z_x be the set of adjacent vertices of x in \mathfrak{D}_f , then the set of adjacent vertices of x in $\overline{\mathfrak{D}_f}$ is $(Z_x \cup \{x\})^c$, that is open in $\mathfrak{T}_{\overline{\mathfrak{D}_f}}$. Since $\mathfrak{T}_{\overline{\mathfrak{D}_f}} = \mathfrak{T}_{\mathfrak{D}_f}^c$, we have $(Z_x \cup \{x\})^c \in \mathfrak{T}_{\overline{\mathfrak{D}_f}}$ and so $Z_x \cup \{x\} \in \mathfrak{T}_{\mathfrak{D}_f}$. therefore, $U_x \subseteq Z_x \cup \{x\}$, where U_x is the smallest open set containing x in $\mathfrak{T}_{\mathfrak{D}_f}$. Hence $U_x = \{x\}$, because $U_x \subseteq Z_x^c$, thus $(V, \mathfrak{T}_{\mathfrak{D}_f})$ is discrete.

The following example shows, the fuzzy non-adjacency compatible vertices topology of the complement $\overline{\mathfrak{D}_f}$ of a graph \mathfrak{D}_f with discrete fuzzy non-adjacency compatible vertices topology can have also discrete fuzzy topology.

Example 1: Suppose $\mathfrak{D}_f = C_5^c$ then $\overline{\mathfrak{D}_f} = C_5$ and both of them have discrete fuzzy topology.

4. Conclusions

This study successfully introduced and analyzed the main and important properties of fuzzy non-adjacency compatible vertices topological spaces associated with fuzzy directed graphs. Through rigorous investigation and the establishment of foundational propositions, corollaries, and some examples, this research clarified several important topological characteristics, such as compactness, Alexandroff space properties, and conditions for discreteness. The findings significantly contribute to the theoretical enrichment of fuzzy graph theory and its topological dimensions. Future studies could explore the applicability of fuzzy topological structures in addressing real-world problems characterized by inherent uncertainty and complexity.

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