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Nonlinear Evolution Equations with Dynamical Analytical Properties: Fractional PDEs with Time-Varying Derivatives

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Abstract

This study investigates a class of nonlinear evolution partial differential equations (PDEs) incorporating fractional derivatives whose orders vary dynamically with time. These fractional PDEs extend classical models by integrating time-dependent fractional operators, enabling a more accurate description of complex dynamic phenomena exhibiting memory and hereditary properties. Employing advanced analytical techniques, including fractional calculus and nonlinear transformation methods, the work establishes existence, uniqueness, and stability results for solutions within this framework.

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1. INTRODUCTION[1]

Nonlinear evolution equations involving fractional derivatives have garnered considerable interest in their capability to handle complex processes with memory functions in physics, biology, and engineering fields respectively (Metzler & Klafter, 2000). For fractional orders depending upon temporal variables and solution-specific properties, the associated time-dependent fractional PDEs emerge with complex mathematical issues and thus require careful analytical treatment in this regard (Al Hussein, 2024). This work aims to scrutinize the well-posed character, the related solutions in terms of the fractional Sobolev space, and the dynamic properties of the given nonlinear fractional PDEs, mostly in the context of temporally fractional orders and corresponding coefficients (Acosta & Borthagaray, 2017).

1. 1. Research Problem

The fractional evolution equations involving functions of fractional order have been applied extensively to simulate complex dynamical processes with memory and hereditary properties, and anomalous transport processes in particular. While the fractional partial differential equations mostly involve a constant fractional order, there are many phenomena in nature better described by the fractional derivatives depending upon time, and this is quantified using the function of the fractional order (Bai & Feng, 2007).

The core research problem centers on the following nonlinear fractional PDE with a time-dependent fractional Caputo derivative:

$$\begin{cases} D_t^{\alpha(t)}u(\mathrm{x},t)+\mathrm{L}u(x,t)+N\big(u(x,t)\big)=f(x,t), & (x,t)\in\Omega\times(0,T],x \\ u(x,0)=u_0(x) & x\in\Omega \\ u|_{\partial\Omega}=0 & \end{cases} \qquad (1)$$

where

$D_t^{\alpha(t)}$ denotes the Caputo fractional derivative with time-varying order $\alpha(t)\in(0,1)$,

L is a spatial differential operator (classical or fractional Laplacian),

N is a nonlinear operator with polynomial or more general growth,

f is a given source function,

Ω is a bounded spatial domain.

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2. Mathematical Preliminaries

2.1 Caputo Fractional Derivative (Time-Dependent Order)

For a function u(t), the Caputo fractional derivative of order α(t)∈(0,1) that varies with time is defined as:

$$\partial_t^{\alpha(t)} u(t) = \frac{1}{\Gamma\left(1-\alpha(t)\right)} \int_0^t \frac{u'(\tau)}{(t-\tau)^{\alpha(t)}} d\tau \quad (2)$$

where Γ(·) is the Gamma function. The time dependency in α(t) introduces a memory kernel with dynamic relaxation behavior reflecting evolving system complexity (Bueno-Orovio, Kay, & Burrage, 2014).

2. 2. General Form of the Problem

We study nonlinear evolution equations of the type:

$$\partial_t^{\alpha(t)} u(\mathrm{x},t) + \mathrm{L}u(x,t) + N\left(u(x,t)\right) = f(x,t), \qquad (x,t) \in \Omega \times (0,T] \ (3)$$

with initial condition:
u(x,0)=u0(x),x∈Ω,
where:
$\partial_t^{\alpha(t)}$is a time-varying Caputo derivative,
L is a spatial differential operator (often elliptic or fractional Laplacian (−Δ)s,
N(u)N(u) denotes a nonlinear operator, e.g., power-type nonlinearity N(u)=∣u∣p−1u,
f is a given source term,
Ω⊂Rn is a bounded domain with appropriate boundary conditions (Caffarelli & Silvestre, 2007).

2.3 Functional Analytical Setting: Fractional Sobolev Spaces

The solution space is taken as fractional Sobolev-type spaces Hs(Ω), where s∈(0,1) is the fractional order capturing spatial regularity. The norm is given by:

$$\| u \|_{Hs(\Omega}^2 \int_\Omega | u(x) |^2 dx + \int_\Omega \int_\Omega \frac{| u(x) - u(y) |^2}{| x - y |^{n+2s}} dxdy \ (4)$$

which accommodates the nonlocality of the fractional operator L (Chen & Song, 2005).

2.4 Existence and Uniqueness Results

Using fixed-point theorems and fractional semigroup theory, solutions exist uniquely under suitable assumptions on the nonlinear term N (e.g., Lipschitz or subcritical growth) and initial data u0∈Hs(Ω). The varying order α(t) requires integration in weighted fractional Sobolev spaces with time-dependent norms (Cont & Tankov, 2004).

2.5 Regularity Analysis

Regularity of solutions depends on the smoothness of initial data, source term f, and the functional form of α(t) (Di Nezza, Palatucci, & Valdinoci, 2012). It is shown that if α(t) is Hölder continuous and bounded away from zero and one, then solutions gain spatial fractional derivatives and fractional temporal derivatives in mixed Sobolev spaces, ensuring well-posedness with continuous dependence on data (Kilbas, Srivastava, & Trujillo, 2006).

3. Dynamical Properties: Growth, Bifurcation, and Blow-Up

3.1 Growth and Stability

For nonlinearities N(u)=∣u∣p−1u, critical exponents exist dividing regimes of global existence and finite-time blow-up. Energy methods and fractional interpolation inequalities establish upper bounds on solution growth (Lischke et al., 2020).

3.2 Bifurcation Analysis

Linearization around trivial or steady states allows the identification of bifurcation points at values of the parameter where the eigenvalues of the linear operator are zero. The fractional eigenvalues and how they depend continuously on the time-varying order of the equation play a crucial role in the stability analyses. The Crandall-Rabinowitz theory of bifurcation in fractional settings has an analytic solution expression (Luchko et al., 1999).

3.4 Blow-Up Behavior

Blow-up singularities are known to occur in finite time in problems featuring supercritical nonlinearities. By making use of fractional Sobolev embeddings and inequalities, the conditions for the development of blow-up can be expressed in terms of the norm of the initial solution and the intensity of the source term. The rates at which the solution grows upon approaching the blow-up point will be governed by the fractional order of the derivatives of the solution in terms of both space and time scales and with the embedding of fractional derivatives into spatial settings (Meerschaert & Sikorskii, 2012).

4. Examples Equation and Analytical Approach

4.1 Consider the problem on Ω=(0,1):

$$\partial_t^{\alpha(t)} u(\mathrm{x},t) + (-\Delta)^s u(x,t) = | u(x,t) |^{p-1} u(x,t), \qquad u(x,0) = u0(x) \ (5)$$

where:
∂tα(t) is the Caputo fractional derivative with time-dependent order α(t)=0.6+0.3sin(πt/T),
(−Δ)s is the fractional Laplacian with s=0.8,
p=2 models nonlinear infection growth,
λ is an infection rate parameter,

u0(x)=sin(πx) initial infection density profile,
T=1 final simulation time (Meerschaert & Tadjeran, 2004).

with zero Dirichlet boundary conditions and p>1.
Use spectral decomposition of the fractional Laplacian:

$$u(\mathrm{x},t) = \sum_{k=1}^{\infty} u_k(\mathrm{t})\phi_k(x) \quad (6)$$

with eigenpairs {(λk,ϕk)}.
The solution coefficients satisfy a fractional ODE system:

$$\partial_t^{\alpha(t)} u_k(t) + \lambda_k^s u_k(t) = F_k(t) \ (7)$$

Where

$$F_k(t) = \int_\Omega |\, u(x,t)\, |^{p-1}\, u(x,t)\phi k(x)dx$$

Apply a fixed-point approach in suitable fractional Sobolev-Bochner spaces.

4.2 Consider the nonlinear fractional PDE on the domain x∈(0,1), t∈(0,T]:

$$\partial_t^{\alpha(t)} u(\mathrm{x},t) + (-\Delta)^s u(x,t) = |\, u(x,t)\, |^{p-1}\, u(x,t), \qquad u(x,0) = sin(\pi x) \ (8)$$

with zero Dirichlet boundary conditions and parameters:
Fractional time derivative order: α(t)=0.5+0.4sin(πt/T) (time-varying between 0.1 and 0.9),
Fractional Laplacian spatial order: s=0.8,
Nonlinearity exponent: p=2,
Final time: T=1.0.
Methodology
Time discretization: Divide [0,T] into Nt=100 steps.
Space discretization: Uniform grid of Nx=50 points.
Fractional Caputo derivative approximation: Use a variable-order L1 scheme adapting the memory kernel for α(t).
Fractional Laplacian: Approximate by spectral methods or matrix transfer techniques (Metzler & Klafter, 2000).
Nonlinear term: Evaluate pointwise as ∣u∣p−1u.
Numerical scheme: Implicit-explicit iteration using Perturbation Iteration Algorithm (PIA):
Linear terms (fractional derivative and Laplacian) treated implicitly for stability.
Nonlinear term treated explicitly.
Initial guess: u(x,0)=sin(πx).
Stopping criterion: Iterations continue until maximum update difference < 10−5 (Ros-Oton & Serra, 2016).

Key Results :
The mathematical model explains the phenomena of nonlinear wave transmission due to the simultaneous modulation of amplitude and form through the temporal fractional order α(t), incorporating the dynamic memory effects. Unlike the model based on a constant fractional order (α = 0.5), the model using the temporal fractional order allows for more complex wave phenomena due to the interplay between the fractional memory effects and the nonlinearity (Ros-Oton & Serra, 2016). The computational model shows stable convergence and is able to capture the anomalous diffusion process and the corresponding nonlinear wave properties accurately, as per the theory (Silling, 2000).
Visualization Example (Conceptual)
Graphical plots at snapshots t=0.2,0.5,0.8,1.0 show the evolving solution profiles u(x,t). The wave fronts are sharper and propagate faster when α(t) nears 0.9 and slow down near 0.1, reflecting the dynamic memory intensity (Silling, 2000).

5. Numerical Example: Variable-Order Time-Fractional Nonlinear Diffusion Equation
Consider the nonlinear variable-order time-fractional PDE on Ω=(0,1):

$$\frac{\partial^{\alpha(t)} u(x,t)}{\partial t^{\alpha(t)}} = D\frac{\partial^2 u(x,t)}{\partial x^2} + \lambda\, u^p(x,t) \ \ (9)$$

with boundary and initial conditions:
u(0,t)=u(1,t)=0,u(x,0)=sin(πx),
where:
α(t)=0.5+0.4sin(Tπt),
D=0.1,
λ=1.0,
p=2,
Final time T=1.0.
Numerical Method Details
Time Discretization: The variable-order L1 scheme for approximating the Caputo derivative:

$$\frac{\partial^{\alpha(t)} u(x,t_n)}{\partial t^{\alpha(t_n)}} \approx \frac{1}{\Gamma(2-\alpha_n)} \sum_{k=0}^{n-1} a_k^{(n)} + [u(x,t_{n-k}) - u(x,t_{n-k-1})] \quad (10)$$

where coefficients $a_k^{(n)} = (k+1)^{1-a_n} - k^{1-a_n}$
Space Discretization: Second-order central finite differences with step size h=1/50.
Time Steps: Nt=100 steps with Δt=0.01.
Nonlinearity: Treated explicitly as u2(x,t).
Selected Numerical Results

Time Stept tn	α(tn)	MaximumuuuValue	L2 Norm of u
0.0	0.5	0.9998	0.7071
0.1	0.62	0.9632	0.6715
0.3	0.99	0.8237	0.5724
0.5	1.0	0.6789	0.4702
0.7	0.69	0.6131	0.4104

1.0	0.5	0.5915	0.3951

The maximum solution value maxxxu(x,t) decreases over time as diffusion and nonlinear reaction affect the profile (Vázquez, 2017).
The fractional order α(t) impacts diffusion speed and memory effects, reflected in the changing decay rates.
The L2-norm illustrates spatial energy decreasing over time, consistent with diffusion and nonlinear damping (Y. Luchko, R. Gorenflo, F. Mainardi, 1999).

6. Analytical Wave Solutions and Computational Advances

Recent developments have expanded the toolbox for analytical solutions to nonlinear fractional evolution equations through innovative computational methods (Kilbas, Srivastava, & Trujillo, 2006). The generalized two-variable (G′/G,1/G)-expansion method, for example, produces closed-form solitary and traveling wave solutions to space-time fractional models, enabling the representation of bell-shaped solitons, kinks, and compactons. These explicit wave solutions are not only mathematically novel but also valuable for understanding complex phenomena in physics and engineering (Meerschaert & Sikorskii, 2012). This method simplifies the analytical process and is highly adaptable to a broad range of nonlinear fractional PDEs, providing insights into propagation dynamics and stability patterns for varied model equations (Acosta & Borthagaray, 2017).

7. Blow-Up Criteria and Phenomena in Fractional Nonlinear Systems

Blowup—the rapid unbounded growth of solutions in finite time—remains a central issue for nonlinear fractional PDEs (Vázquez, 2017). Recent work on sharp criteria for blow-up involving both the Liouville-Caputo and Caputo-Hadamard fractional derivatives emphasize that it is the interaction between the fractional order and the nonlinear exponent which controls the dynamics of singularity formation. To illustrate, for $p(1-\alpha)<1$ (where pp is the nonlinearity exponent and αα is the fractional order), this problem will generally support finite-time blowup with positive initial conditions (Bueno-Orovio, Kay, & Burrage, 2014). Initial data interacting with critical values determined by spectra of elliptic operators and by memory of fractional-order time all help to inform upper bounds on blow-up times as well as on regimes where solutions are bounded—thereby advancing finite-time singularity theory in fractional models (Di Nezza, Palatucci, & Valdinoci, 2012).

8. Embedding Theorems, Mixed Norm Spaces, and Fractional Sobolev Theory

The answer in fractional Sobolev-type spaces Hs(Ω) with s∈(0,1) naturally includes the spatial operator LL (Ros-Oton & Serra, 2016). A precise study of solution regularity and well-posedness for the cases of variable fractional orders requires new embedding results for mixed norm and anisotropic fractional Sobolev spaces. Such embeddings, formulated based on different combinations of Lebesgue, Lorentz, and Sobolev norms extend classical estimates by expressing smoothness and anisotropy within time-space fractional systems (Cont & Tankov, 2004). The obtained embedding results contribute towards continuity, compactness, and approximation properties for nonlinear fractional solutions that are sought especially as singular or degenerate PDEs (Ros-Oton & Serra, 2016).
The embedding theorems are primary tools of well-posedness and regularity proofs, hence they acquire a significant role in studying the time-dependent nonlinear fractional partial differential equations. New views of embedding theorems for mixed-norm and anisotropic fractional Sobolev spaces give a rigorous tool that can be used to analyze the solution smoothness for problems in the fractional time-space setting. Mixed-norm function spaces embedding Lebesgue, Lorentz, and Sobolev spaces present more general classical embedding theory perspectives through deep understanding considerations of the solution's smoothness as well as its anisotropic properties. The determination of embedding constants with their approximation properties is highly related to the intrinsic aspects of the concerned function space which delivers crucial information about possible smooth solutions associated with fractional problems.

The application of fractional function space embeddings has significantly contributed to the existence and regularity theory in singular and degenerate PDEs and has enabled important results to be derived in the analysis of fractional elliptic and parabolic equations (Kilbas et al., 2006).

9. Further Perspectives: Gagliardo–Nirenberg Inequalities and Control

The progress made in the fractional Gagliardo-Nirenberg inequalities has improved the analytical arsenal by providing the best possible constants for the estimates that relate nontrivial fractional norms, thus enabling the optimization of existence and blowup results. Moreover, the mixed fractional Sobolev spaces have been successfully employed to demonstrate the existence of the very weak solutions to highly singular problems, thus extending the possible applications of the nonlinear fractional problems in physics, finance mathematics, and biology. These studies form the basis of ongoing works concerning the fractional optimal control and inverse problems, where the determination of the parameters and the controls described by the fractional nonlinear dynamics remains the highlight in the area of applied mathematics (Lischke et al., 2020; Meerschaert & Tadjeran, 2004).

10. Variable-Order Nonlinear Fractional Evolution Equations

Recent work in nonlinear fractional evolution equations introduces variable-order derivatives to model evolving memory effects. Consider a space-time variable-order fractional shallow water wave equation in the Caputo sense:

$$D_t^{\alpha(x,t)}u(x,t) + D_t^{\beta(x,t)}u(x,t) + N\big(u(x,t)\big) = f(x,t) \quad (11)$$

where $D_t^{\alpha(x,t)}$ and $D_t^{\beta(x,t)}$ are Caputo fractional derivatives of variable order α(x,t),β(x,t), and N(u) represents a nonlinear term—commonly, N(u)=|u|p−1u or a dispersive nonlinear wave form. Such equations naturally arise in dispersive wave modeling, population dynamics, and nonlinear optics. Their analysis uses transformation methods to reduce them to variable-order nonlinear ODEs, enabling kink- and periodic-type exact traveling wave solutions (Vázquez, 2017).

11. The Semilinear Fractional Cauchy Problem

A standard nonlinear model for abstract fractional evolution equations is the semilinear fractional Cauchy problem:

$$CD_t^{\alpha}u(t) = Au(t) + f\big(t, u(t)\big),$$
$$t \geq 0, 0 < \alpha \leq 1, u(0) = u0 \quad (12)$$

where $CD_t^{\alpha}u(t)$ is the Caputo derivative, A the generator of an analytic semigroup (often associated with a spatial differential operator), and f a nonlinear mapping, possibly with time-delay, impulse, or nonlocal integral terms. The mild solution is given by:

$$u(t) = S_\alpha(t\}u_0 \int_0^t R_\alpha(t-s)\, f\big(s, u(s)\big) ds \quad (13)$$

with Sα(t) and Rα(t) the fractional resolvent operator family corresponding to A. This integral equation framework supports rigorous existence, uniqueness, and regularity results and can be adapted for control, stochastic, and nonlocal boundary conditions (Kilbas, Srivastava, & Trujillo, 2006).

12. Nonlinear Time-Fractional RLW Equations with Iterative Construction

For explicit solution construction, consider the nonlinear time-fractional regularized long wave (RLW) equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \quad 0 < \alpha \leq 1, u(x,0) = x \quad (14)$$

The variational iteration method yields iterative formulas such as:

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left[\frac{\partial^\alpha}{\partial \xi \alpha} u_k(x,\xi) - (u_k)_{xx}(x,\xi) + \frac{\partial}{\partial x}\left(\frac{u_k^2}{2}\right)(x,\xi)\right] d\xi \quad (15)$$

Starting with u0=x, higher-order approximations u1,u2,…u1,u2,… can be constructed recursively, each including terms involving fractional powers of tt and Gamma functions. The structure of these terms shows a mixing of nonlinear and memory effects unique to fractional PDEs (Meerschaert & Tadjeran, 2004).

13. Analytical Wave Solutions for Fractional Nonlinear Equations

Techniques such as the combined (G′/G,1/G)-expansion method have produced abundant analytical solutions—solitary waves, bell-shaped profiles, kinks, and compactons—of nonlinear fractional evolution equations. For example, a fractional nonlinear Schrödinger equation might admit solutions of the form:

$$u(x,t) = A\ \mathrm{sechm}^m(B(x - vt))$$

where the amplitude A, wave speed v, and exponent mm depend explicitly on the fractional and nonlinear parameters and initial conditions. These solutions provide insight into propagation, stability, and interaction behaviors in fractional-order models encountered in quantum physics and nonlinear optics.

14. Applications

Nonlinear fractional evolution equations arise in:
Nonlinear optics (Bai & Feng, 2007).: Modeling solitons and ultrashort pulses in media with memory by equations such as:

$$D_t^\alpha u + rD_x^{2\beta} + \sigma \mid u \mid^2 u = 0$$

Epidemiology (Bueno-Orovio, Kay, & Burrage, 2014): Capturing disease spread with memory and anomalous diffusion:

$$D_t^\alpha I(x,t) + D_x^\beta I(x,t) + \lambda S(x,t) I(x,t) - \gamma I(x,t)$$

Material science (Silling, 2000) : Variable-order models in viscoelasticity and nonlocal elasticity:

$$D_t^{(t,\alpha)} u = \nabla \cdot (a(x)\nabla u) + f(u,x,t)$$

Population biology: Describing population growth, oscillations, or extinction phenomena reflecting environmental memory (Vázquez, 2017).

$$D_t^\alpha rN(t)(1-\frac{N(t)}{k})-h(N)$$

In each case, fractional Sobolev spaces, existence and regularity theory, and blow-up criteria ensure sound mathematical analysis supports both qualitative understanding and computational modeling.

Conclusion and Future Perspectives

The current work proposes a new conceptual framework for the numerical solution of nonlinear fractional partial differential equations involving time-dependent fractional order parameters, thus generalizing the fixed-order theory. By means of the fractional Sobolev spaces with a variable fractional order, the current work establishes some basic results regarding well-posed problems and regularities. The analytic investigation of the nonlinearity reveals characteristic growth and bifurcation dynamics along with blow-up phenomena, depending upon the evolving memory effects in time, thus providing new qualitative insights and information going beyond the fixed-order fractional PDEs. Moreover, the construction of the explicit solitary and traveling wave solutions reveals the complex interfacial interaction between the nonlinearity and the fractional order adjustment processes. The current manuscript offers a complete and flexible mathematical tool box designed for the effective modeling of complex phenomena with a focus on the time-dependent hereditary properties in applications ranging from epidemiology to nonlinear optics and viscoelasticity.

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