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# A Novel Subordination-Based Class of Bi-Univalent Functions Related to Lucas–Balancing Polynomials

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*A b s t r a c t*

In this paper, new subclasses of analytic bi-univalent functions in the open unit disk are introduced by means of differential operators associated with Lucas–Balancing polynomials. These subclasses are defined through suitable subordination conditions imposed on both the function and its inverse. Estimates for the initial Taylor–Maclaurin coefficients are obtained, particularly for $|a_2|$ and $|a_3|$. In addition, the Fekete–Szegö functional is investigated and corresponding inequalities are established.



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## 1. INTRODUCTION

Let $\mathcal{A}$ stands the class of functions Ƒ which are holomorphic in the open unit disk $\tilde{E}$, $\tilde{E} = \{z : z \in C$ and $|z|<1\}$.

Every function Ƒ can be represented by the Taylor series:

$$Ƒ(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

It is clear that the function Ƒ satisfies the normalized conditions $Ƒ(0) = Ƒ'(0) - 1 = 0$.

Moreover, Let $\Upsilon(z)$ be holomorphic function in $\tilde{E}$ of the form:

$$\Upsilon(z) = k_{\circ} + k_1 z + k_2 z^2 + k_3 z^3$$

The class S consists of all functions Ƒ that are holomorphic and univalent in the unit disk $\tilde{E}$. By virtue of the Koebe one-quarter theorem, the image of $\tilde{E}$ under any function $Ƒ \in S$ contains a disk whose radius is at least $\frac{1}{4}$ (Duren, 1983).

As a consequence, every univalent function Ƒ, has an inverse function $Ƒ^{-1}$ such that:

$$Ƒ^{-1}\left(Ƒ(z) = z(z \in \tilde{E})\right)$$

and

$$\left(Ƒ^{-1}(\omega)\right) = \omega, \left(|\omega| < r_{\circ}(Ƒ) \geq \frac{1}{4}\right), \qquad (2)$$

A function Ƒ is called bi-univalent in the unit disk $\tilde{E}$ if both Ƒ and $Ƒ^{-1}$ are univalent in $\tilde{E}$. The family of all such functions is denoted by Σ.

If $Ƒ \in \Sigma$ is given in the normalized form described above, then its inverse function $g = Ƒ^{-1}$ admits the series expansion:

$$g(\omega) = Ƒ^{-1}(\omega) = \omega - a_2\,\omega^2 + (2a_2^2 - a_3)\omega^3 - \cdots. \qquad (3)$$

Lewin (1967) introduced the class of bi-univalent functions and established the bound $|a_2| \leq 1.51$. Later, Brannan & Clunie (1980) conjectured that $|a_2| \leq \sqrt{2}$, and Netanyahu in (Ma & Minda, 1992) proved that $\max_{Ƒ\in\Sigma}|a_2| = \frac{4}{3}$.

Sinc then, several different subclasses of class Σ have been investigated, with estimates for the coefficients $|a_2|$ and $|a_3|$ provided by multiple authors (Ali et al., 2012; Brannan & Taha, 1988; Goodman, 1983;

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Juma & Aziz, 2012; Robertson, 1970; Saloomi et al., 2024; Srivastava et al., 2010).

Among these developments, two subclasses of Σ have been defined using the concept of quasi-subordination.

In (Ma & Minda, 1992), Ma and Minda introduced a unified approach through which many geometric properties can be systematically derived. Within this framework, they defined two subclasses of analytic functions, namely the Ma–Minda starlike class and the Ma–Minda convex class, as follows:

$$S*(\phi) = \{ Ƒ \in \mathcal{A}: (\frac{zƑ'(z)}{Ƒ(z)}) \prec \phi(z);\ z \in \tilde{E}\}$$

$$K(\phi) = \left\{ Ƒ \in \mathcal{A}: 1 + \frac{zƑ''(z)}{Ƒ'(z)} \prec \phi(z);\ z \in \tilde{E} \right\}$$

Here, ϕ is an analytic and univalent function in the open unit disk Ẽ with positive real part, satisfying the normalization conditions: $\phi(0) = 1$ and $\phi'(0) > 0$.

Moreover, the image domain $\varphi(\tilde{E})$ is starlike with respect to point 1 and is symmetric respect to the real axis.

The class S ∗ (ϕ) is referred to as the Ma–Minda starlike class, while $K(\phi)$ is known as the Ma–Minda convex class.

Let Ƒ and $g$ be analytic in the unit disk Ẽ. We say that Ƒ is subordinate to $g$, written $Ƒ(z) \prec g(z)$, if there exists an analytic function $\omega: \tilde{E} \to \tilde{E}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $Ƒ(z) = g(\omega(z))$ (Juma & Saloomi, 2018; Miller & Mocanu, 1981; Saloomi et al., 2021).

This work is devoted to introducing and studying a newly defined subclass of bi-univalent functions in the open unit disk. The subclass is formulated within the framework of analytic subordination, where Lucas–Balancing polynomials play a fundamental role in determining the defining conditions. In this setting, both the function and its inverse are required to satisfy analogous subordination relations, which preserves the structural properties necessary for the study of bi-univalent functions.

An essential aspect of the analysis concerns the coefficient behavior of functions belonging to this subclass. In particular, emphasis is placed on determining bounds for the initial Taylor–Maclaurin coefficients, especially the coefficients $|a_2|$ and $|a_3|$. The obtained estimates are derived by adapting classical techniques from geometric function theory, including the use of Schwarz functions and coefficient comparison methods, to the polynomial-based subordination framework. These findings not only yield useful bounds for the early coefficients but also contribute to a clearer understanding of the analytic structure and geometric behavior of functions within this class.

### 1. 1. Lucas -Balancing Polynomials

The Lucas-Balancing polynomials defined the following recurrence relation:

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x)$$

Where $C_\circ(x) = 1$ and $C_1(x) = 3x$ and $C_2(x) = 18x^2 - 1$ and:

$$G(z) = \sum_{n=2}^{\infty} B_n(x)\, z^n = \frac{1 - 3xz}{1 - 6xz + z^2}$$

Where $G(z)$ is the generating function.

## 2. THE CLASS $\mathcal{M}_{\alpha,\theta}^{\curlyvee}$ (Ƒ)(z) AND $\mathcal{M}_{\alpha}$(Ƒ)(z)

**Definition 2.1**

Consider a function Ƒ belong in to the class $\mathcal{M}_{\alpha,\theta}^{\curlyvee}$ (Ƒ)(z) as defined in Equation (1). where the parameters satisfy: $0 \le \alpha \le 1, \curlyvee \ge 1, \frac{1}{2} < x \le 1$ and $\theta \epsilon \mathbb{C}\backslash\{0\}$, if and only if:

$$1 + \frac{1}{\theta}\left(\left[v\frac{\left((zƑ'(z))'\right)^{\curlyvee}}{Ƒ'(z)} + (1-v)(\frac{\left((zƑ'(z))\right)^{\curlyvee}}{Ƒ(z)}) - 1\right] + \left(\alpha(\frac{\left((zƑ'(z))\right)^{\curlyvee}}{Ƒ(z)}) - \alpha\right)\right) \prec R(x, z), (z \in \tilde{E}) \quad (4)$$

$$1 + \frac{1}{\theta}\left(\left[v\frac{\left((\omega\ g'(\omega))'\right)^{\curlyvee}}{g'(\omega)} + (1-v)(\frac{\left((\omega\ g'(\omega))\right)^{\curlyvee}}{g(\omega)}) - 1\right] + \left(\alpha(\frac{\left((\omega\ g'(\omega))\right)^{\curlyvee}}{g(\omega)}) - \alpha\right)\right) \prec R(x, z), (\omega \in \tilde{E}) \quad (5)$$

**Theorem 2.1**

Let $Ƒ \in \Sigma$ belong to $\mathcal{M}_{\alpha,\theta}^{\tau}$ (Ƒ)(z), $0 \le \alpha \le 1$ if it fits:

$$|k_2| \le \frac{3|\theta|\sqrt{3}|x|\sqrt{|x|}}{\sqrt{(9x^2)\theta[\curlyvee(\curlyvee-2)(3v+\alpha+1) + \curlyvee(\curlyvee+1)(\alpha+1) + v(3\curlyvee^2+1)] - (2\curlyvee-1)^2(v+\alpha+1)^2(18x^2-1)}}$$

$$|k_3| \le \frac{\left|1 - \frac{(2\curlyvee^2 - 4\curlyvee + 1)\alpha}{2(3\curlyvee-1)(2v+\alpha+1)}(27x^3)\theta^2\right|}{(9x^2)\theta[\curlyvee(\curlyvee-2)(3v+\alpha+1) + \curlyvee(\curlyvee+1)(\alpha+1) + v(3\curlyvee^2+1)] - (2\curlyvee-1)^2(v+\alpha+1)^2(18x^2-1)} + \frac{|3x||\theta|}{|(3\curlyvee-1)(2v+\alpha+1)|}$$

**Proof:**

Since $\digamma \in \mathcal{M}_{\alpha,\theta}^{\gamma}\ (\digamma)(z)$, there exist holomorphic functions $\mu: \hat{E} \to \hat{E}$, such that the following relations are satisfied where:

$$u(z) = s_1 z + s_2 z^2 + \cdots \text{ such that } |u(z)| \le 1 \tag{6}$$

$$v(\mathrm{w}) = n_1 \mathrm{w} + n_2 \mathrm{w}^2 + \cdots \text{ such that } |v(\mathrm{w})| \le 1 \tag{7}$$

$$1+\frac{1}{\theta}\left(\left[v\frac{\left((z\digamma'(z))'\right)^{\gamma}}{\digamma'(z)}+(1-v)\left(\frac{\left((z\digamma'(z))\right)^{\gamma}}{\digamma(z)}\right)-1\right]+\left(\alpha\left(\frac{\left((z\digamma'(z))\right)^{\gamma}}{\digamma(z)}\right)-\alpha\right)\right)=\mathrm{R}(\mathrm{x},\mathrm{u}(z)),\ (z\in\tilde{E}) \tag{8}$$

$$1+\frac{1}{\theta}\left(\left[v\frac{\left((\omega\ g'(\omega))'\right)^{\gamma}}{g'(\omega)}+(1-v)\left(\frac{\left((\omega\ g'(\omega))\right)^{\gamma}}{g\ (\omega)}\right)-1\right]+\left(\alpha\left(\frac{\left((\omega\ g'(\omega))\right)^{\gamma}}{g\ (\omega)}\right)-\alpha\right)\right)=\mathrm{R}(\mathrm{x},\mathrm{v}(\mathrm{w})),\ (\mathrm{w}\in\tilde{E}) \tag{9}$$

Such that:

$$1+\frac{1}{\theta}\left(\left[v\frac{\left((z\digamma'(z))'\right)^{\gamma}}{\digamma'(z)}+(1-v)\left(\frac{\left((z\digamma'(z))\right)^{\gamma}}{\digamma(z)}\right)-1\right]+\left(\alpha\left(\frac{\left((z\digamma'(z))\right)^{\gamma}}{\digamma(z)}\right)-\alpha\right)\right)=1+\frac{1}{\theta}\left((1+\alpha)k_2 z+[(2+4\alpha)k_3-(1+3\alpha)k_2^2]z^2+\cdots\right) \tag{10}$$

$$1+\frac{1}{\theta}\left(\left[v\frac{\left((\omega\ g'(\omega))'\right)^{\gamma}}{g'(\omega)}+(1-v)\left(\frac{\left((\omega\ g'(\omega))\right)^{\gamma}}{g\ (\omega)}\right)-1\right]+\left(\alpha\left(\frac{\left((\omega\ g'(\omega))\right)^{\gamma}}{g\ (\omega)}\right)-\alpha\right)\right)=1+\frac{1}{\theta}\left(-(1+\alpha)k_2\mathrm{w}+[(3+5\alpha)k_2^2-(2+4\alpha)k_3]\mathrm{w}^2+\cdots\right) \tag{11}$$

From Equations (10) and (11) we get:

$$(2\gamma-1)(v+\alpha+1)k_2=\theta R_1(x)s_1 \tag{12}$$

$$(3\gamma-1)(2v+\alpha+1)k_3+(2\gamma^2-4\gamma+1)(3v+\alpha+1)k_2^2=\theta R_1(x)s_2+\theta R_2(x)s_1^2 \tag{13}$$

$$-(2\gamma-1)(v+\alpha+1)k_2=\theta R_1(x)n_1 \tag{14}$$

$$[(2\gamma^2-2\gamma-1)(\alpha+1)+(6\gamma^2-1)]k_2^2-(3\gamma-1)(2v+\alpha+1)k_3=\theta R_1(x)n_2+\theta R_2(x)n_1^2 \tag{15}$$

From Equations (12) and (14):

$$s_1=-n_1 \tag{16}$$

$$(2\gamma-1)^2(v+\alpha+1)^2k_2^2=\theta^2R_1^2(x)s_1^2$$

$$(2\gamma-1)^2(v+\alpha+1)^2k_2^2=\theta^2R_1^2(x)n_1^2$$

$$2(2\gamma-1)^2(v+\alpha+1)^2k_2^2=\theta^2R_1^2(x)(s_1^2+n_1^2)$$

FromEquations (13), (15) and (16):

$$[(2\gamma^2-4\gamma+1)(3v+\alpha+1)+(2\gamma^2-2\gamma-1)(\alpha+1)+(6\gamma^2-1)v]k_2^2=\theta R_1(x)(s_2+n_2)+\theta R_2(x)(s_1^2+n_1^2$$

$$k_2^2=\frac{\theta^2R_1^3(x)(s_2+n_2)}{2[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)]\theta R_1^2(x)-2(2\gamma-1)^2(v+\alpha+1)^2R_2(x)} \tag{17}$$

$$|k_2|\le\frac{3\theta\sqrt{3}|x|\sqrt{|x|}}{\sqrt{(9x^2)\theta[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)]-(2\gamma-1)^2(v+\alpha+1)^2(18x^2-1)}}$$

From Equations (13), (15) and (17):

$$2(3\gamma-1)(2v+\alpha+1)k_3+(2\gamma^2-4\gamma+1)\alpha-2(3\gamma-1)(2v+\alpha+1)k_2^2=\theta R_1(x)(s_2-n_2)+\theta R_2(x)(s_1^2-n_1^2)$$

$$k_3=\left(1-\frac{(2\gamma^2-4\gamma+1)\alpha}{2(3\gamma-1)(2v+\alpha+1)}\right)k_2^2+\frac{\theta R_1(x)(s_2+n_2)}{2(3\gamma-1)(2v+\alpha+1)} \tag{18}$$

$$|k_3|\le\frac{\left|1-\frac{(2\gamma^2-4\gamma+1)\alpha}{2(3\gamma-1)(2v+\alpha+1)}(27x^3)\theta^2\right|}{(9x^2)\theta[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)]-(2\gamma-1)^2(v+\alpha+1)^2(18x^2-1)}+\frac{|3x||\theta|}{|(3\gamma-1)(2v+\alpha+1)|}$$

**Definition 2.2**

Consider a function $\digamma$ belongs to the class $\mathcal{M}_{\alpha}(\digamma)(z), 0 \leq \alpha \leq 1$ as defined in (1). We say that $\digamma$ is a member of the subclass $\mathcal{M}_{\alpha}(\digamma)(z)$, where the parameters satisfy $0 \leq \alpha \leq 1$, if and only if:

$$\left[(1-\alpha)\frac{z\digamma'(z)}{\digamma(z)} + \alpha(1+\frac{z\digamma''(z)}{z\digamma'(z)})\right] \prec \mathrm{R}(x,z), (z \in \tilde{E}) \quad (19)$$

$$\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha(1+\frac{wg''(w)}{wg'(w)})\right] \prec \mathrm{R}(x,w), (w \in \tilde{E}) \quad (20)$$

**Remark 2.1**

Setting $\alpha = 0$ in the class $\mathcal{M}_{\alpha}(\digamma)(z)$, we obtain Theorem 2.2 of [5].

**Remark 2.2**

By taking $\alpha = 1$ in the class $\mathcal{M}_{\alpha}(\digamma)(z)$, we obtain Theorem 2.1 of [5].

**Theorem 2.2**

Let $\digamma \in \Sigma$ and suppose that $\digamma$ belong to class $\mathcal{M}_{\alpha}(\digamma)(z), 0 \leq \alpha \leq 1$ if satisfy that:

$$|k_2| \leq \frac{3\sqrt{3}|x|\sqrt{|x|}}{\sqrt{(1+\alpha)(9x^2) - (1+\alpha)^2(18x^2-1)}}$$

$$|k_3| \leq \frac{3x}{(2+4\alpha)} + \frac{9x^2}{(1+\alpha)^2}$$

**Proof:**

Since $\digamma$ belong to class $\mathcal{M}_{\alpha}(\digamma)(z)$, there exist holomorphic functions $u, v: \hat{E} \to \hat{E}$, such that:

$$u(z) = s_1 z + s_2 z^2 + \cdots \textit{such that } |u(z)| \leq 1 \quad (21)$$

$$v(w) = n_1 w + n_2 w^2 + \cdots \textit{such that } |v(w)| \leq 1 \quad (22)$$

$$\left[(1-\alpha)\frac{z\digamma'(z)}{\digamma(z)} + \alpha(1+\frac{z\digamma''(z)}{z\digamma'(z)})\right] = \mathrm{R}(x,u(z)), (z \in \tilde{E}) \quad (23)$$

$$\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha(1+\frac{wg''(w)}{wg'(w)})\right] = \mathrm{R}(x,v(w)), (w \in \tilde{E}) \; \blacksquare \quad (24)$$

Such that:

$$\left[(1-\alpha)\frac{z\digamma'(z)}{\digamma(z)} + \alpha(1+\frac{z\digamma''(z)}{z\digamma'(z)})\right] = 1 + (1+\alpha)k_2 z + [(2+4\alpha)k_3 - (1+3\alpha)k_2^2]z^2 + \cdots \quad (25)$$

$$\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha(1+\frac{wg''(w)}{wg'(w)})\right] = 1 - (1+\alpha)k_2 w + [(3+5\alpha)k_2^2 - (2+4\alpha)k_3]w^2 + \cdots \quad (26)$$

and

$$\mathrm{R}(x,u(z)) = 1 + R_1(x)s_1 z + [R_1(x)s_2 + R_2(x)s_1^2]z^2 + \ldots \quad (27)$$

$$\mathrm{R}(x,v(w)) = 1 + R_1(x)n_1 w + [R_1(x)n_2 + R_2(x)n_1^2]w^2 + \ldots \quad (28)$$

$$1 + (1+\alpha)k_2 z + [(2+4\alpha)k_3 - (1+3\alpha)k_2^2]z^2 + \cdots = 1 + R_1(x)s_1 z + [R_1(x)s_2 + R_2(x)s_1^2]z^2 + \cdots \quad (29)$$

$$1 - (1+\alpha)k_2 w + [(3+5\alpha)k_2^2 - (2+4\alpha)k_3]w^2 + \cdots = 1 + R_1(x)n_1 w + [R_1(x)n_2 + R_2(x)n_1^2]w^2 + \cdots \quad (30)$$

From Equations (29) and (30) we get:

$$(1+\alpha)k_2 = R_1(x)s_1 \quad (31)$$

$$(2+4\alpha)k_3 - (1+3\alpha)k_2^2 = R_1(x)s_2 + R_2(x)s_1^2 \quad (32)$$

$$-(1+\alpha)k_2 = R_1(x)n_1 \quad (33)$$

$$(3+5\alpha)k_2^2 - (2+4\alpha)k_3 = R_1(x)n_2 + R_2(x)n_1^2 \quad (34)$$

From Equations (31) and (33):

$$s_1 = -n_1 \quad (35)$$

$$(1+\alpha)^2 k_2^2 = R_1^2(x)s_1^2$$

$$(1+\alpha)^2 k_2^2 = R_1^2(x)n_1^2$$

$$k_2^2 = \frac{R_1^2(x)(s_1^2+n_1^2)}{2(1+\alpha)^2} \quad (36)$$

From Equations (32), (34) and (36):

$$[(3+5\alpha) - (1+3\alpha)]k_2^2 = R_1(x)(s_2+n_2) + R_2(x)(s_1^2+n_1^2)$$

$$k_2^2 = \frac{R_1^3(x)(s_2+n_2)}{2(1+\alpha)R_1^2 - 2(1+\alpha)^2 R_2(x)}$$

$$|k_2| \leq \frac{3\sqrt{3}|x|\sqrt{|x|}}{\sqrt{(1+\alpha)(9x^2) - (1+\alpha)^2(18x^2-1)}}$$

From Equations (32), (34) and (36):

$$2(2+4\alpha)k_3 - [(1+3\alpha) + (3+5\alpha)]k_2^2 = R_1(x)(s_2-n_2) + R_2(x)(s_1^2-n_1^2)$$

$$2(2+4\alpha)k_3 - 4(1+2\alpha)k_2^2 = R_1(x)(s_2-n_2)$$

$$k_3 = \frac{R_1(x)(s_2 - n_2)}{2(2+4\alpha)} + \frac{2R_1^2(x)(s_1^2 + n_1^2)}{2(2+4\alpha)(1+\alpha)} \quad (37)$$

$$|k_3| \le \frac{3x}{(2+4\alpha)} + \frac{9x^2}{(1+\alpha)^2}$$

**Theorem 2.3**

Let $\mathcal{F} \in \Sigma$ and suppose that $\mathcal{F}$ belong to class $\mathcal{M}_{\alpha,\theta}^{\gamma}(\mathcal{F})(z)$ and $\mu \in R$ then:

$$|k_3 - \mu k_2^2| \le \begin{cases} \dfrac{3|x|}{(3\gamma-1)(2v+\alpha+1)} & if\ |1-\mu| \le \mathcal{L} \\ \dfrac{27|x^3|}{(9x^2)[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)] - (2\gamma-1)^2(v+\alpha+1)^2(18x^2-1)} & if\ \ |1-\mu| \ge \mathcal{L} \end{cases}$$

Where:

$$\mathcal{L} = \frac{(9x^2)[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)] - (2\gamma-1)^2(v+\alpha+1)^2(18x^2-1)}{2(3\gamma-1)(2v+\alpha+1)(9x^2)} + \frac{(2\gamma^2-4\gamma+1)\alpha}{2(3\gamma-1)(2v+\alpha+1)}$$

**Proof:**

Since $\mathcal{F}(z) \in \mathcal{M}_\alpha(\mathcal{F})(z)$ and $\mu \in R$ then from Equations (18) and (19) we can write that:

$$k_3 - \mu k_2^2 = \left(1 - \frac{(2\gamma^2-4\gamma+1)\alpha}{2(3\gamma-1)(2v+\alpha+1)}\right)k_2^2 + \frac{R_1^2(x)(s_1^2+n_1^2)}{(2+4\alpha)(1+\alpha)} - \mu k_2^2$$

$$= \frac{R_1(x)(s_2-n_2)}{2(3\gamma-1)(2v+\alpha+1)} + \left(1 - \frac{(2\gamma^2-4\gamma+1)\alpha}{2(3\gamma-1)(2v+\alpha+1)} - \mu\right)k_2^2$$

$$= \frac{R_1(x)(s_2-n_2)}{2(3\boldsymbol{\lambda}-1)(2v+\alpha+1)} + \left(1 - \frac{(2\boldsymbol{\lambda}^2-4\boldsymbol{\lambda}+1)\alpha}{2(3\boldsymbol{\lambda}-1)(2v+\alpha+1)} - \mu\right) \times \frac{R_1^3(x)(s_2+n_2)}{2[\boldsymbol{\lambda}(\boldsymbol{\lambda}-2)(3v+\alpha+1)+\boldsymbol{\lambda}(\boldsymbol{\lambda}+1)(\alpha+1)+v(3\boldsymbol{\lambda}^2+1)]R_1^2(x) - 2(2\boldsymbol{\lambda}-1)^2(v+\alpha+1)^2R_2(x)}$$

Let $\Theta(\alpha,\gamma,\boldsymbol{v}) = 1 - \frac{(2\gamma^2-4\gamma+1)\alpha}{2(3\gamma-1)(2v+\alpha+1)}$ and:

$$\kappa_3 - \mu{\kappa_2}^2 = \frac{R_1(x)(s_2-n_2)}{2(3\gamma-1)(2v+\alpha+1)} + (\Theta(\alpha,\gamma,v)-\mu)k_2^2$$

$$\kappa_3 - \mu{\kappa_2}^2 = R_1(x) \times \Bigg[\Big(\frac{(\Theta(\alpha,\gamma,v)-\mu)R_1^2(x)}{2[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)]R_1^2(x) - 2(2\gamma-1)^2(v+\alpha+1)^2R_2(x)} + \frac{1}{2(3\gamma-1)(2v+\alpha+1)}\Big)s_2 + \Big(\frac{(\Theta(\alpha,\gamma,v)-\mu)R_1^2(x)}{2[\gamma(\gamma-2)(3v+\alpha+1)+\gamma(\gamma+1)(\alpha+1)+v(3\gamma^2+1)]R_1^2(x) - 2(2\gamma-1)^2(v+\alpha+1)^2R_2(x)} - \frac{1}{2(3\gamma-1)(2v+\alpha+1)}\Big)n_2\Bigg]$$

**Theorem 2.4**

Suppose that $\mathcal{F}(z) \in \mathcal{M}_\alpha(\mathcal{F})(z)$ and $\mu \in R$ then:

$$|k_3 - \mu k_2^2| \le \begin{cases} \dfrac{3|x|}{2+4\alpha} & if\ |1-\mu| \le \dfrac{(1-\alpha)(9x^2)-(1+\alpha)^2(18x^2-1)}{(2+4\alpha)(9x^2)} \\ \dfrac{27|x^3|}{(1-\alpha)(9x^2)-(1+\alpha)^2(18x^2-1)} & if\ \ |1-\mu| \ge \dfrac{(1-\alpha)(9x^2)-(1+\alpha)^2(18x^2-1)}{(2+4\alpha)(9x^2)} \end{cases}$$

**Proof:**

Since $\mathcal{F}(z) \in \mathcal{M}_\alpha(\mathcal{F})(z)$ and $\mu \in R$ then from the Equations (18) and (19) we can write that:

$$k_3 - \mu k_2^2 = \frac{R_1(x)(s_2-n_2)}{2(2+4\alpha)} + \frac{R_1^2(x)(s_1^2+n_1^2)}{(2+4\alpha)(1+\alpha)} - \mu k_2^2$$

$$= \frac{R_1(x)(s_2-n_2)}{2(2+4\alpha)} + k_2^2 - \mu k_2^2$$

$$= \frac{R_1(x)(s_2-n_2)}{2(2+4\alpha)} + (1-\mu)k_2^2$$

$$= \frac{R_1(x)(s_2-n_2)}{2(2+4\alpha)} + (1-\mu)\frac{R_1^3(x)(s_2+n_2)}{2(1+\alpha)R_1^2(x) - 2(1+\alpha)^2R_2(x)}$$

$$\kappa_3 - \mu{\kappa_2}^2 = \left\{\frac{(1-\mu)R_1^3(x)}{2\big((1+\alpha)-R_1^2(x)-(1+\alpha)^2R_2(x)\big)} + \frac{R_1(x)}{2(2+4\alpha)}\right\}s_2 + \left\{\frac{(1-\mu)R_1^3(x)}{2\big((1+\alpha)-R_1^2(x)-(1+\alpha)^2R_2(x)\big)} - \frac{R_1(x)}{2(2+4\alpha)}\right\}n_2$$

$$= \left\{\frac{(1-\mu)R_1^3(x)}{2\big((1+\alpha)-R_1^2(x)-(1+\alpha)^2R_2(x)\big)} + \frac{R_1(x)}{2(2+4\alpha)}\right\}s_2 + \left\{\frac{(1-\mu)R_1^3(x)}{2\big((1+\alpha)-R_1^2(x)-(1+\alpha)^2R_2(x)\big)} - \frac{R_1(x)}{2(2+4\alpha)}\right\}n_2$$

$$= \left\{h_1(\mu) + \frac{R_1(x)}{2(2+4\alpha)}\right\} s_2 + \left\{h_1(\mu) - \frac{R_1(x)}{2(2+4\alpha)}\right\} n_2$$

Where $h_1(\mu) = \frac{(1-\mu)R_1^3(x)}{2\left((1+\alpha)-R_1^2(x)-(1+\alpha)^2 R_2(x)\right)}$.

## 3. CONCLUSIONS

In this study, we introduced new subclasses of bi-univalent functions formulated through operators involving Lucas-Balancing polynomial and investigated the associated initial coefficient bounds. By incorporating the structural properties of these polynomials, the proposed subclasses extend previously established families of analytic and bi-univalent functions, thereby enriching the framework within which coefficient estimation problems can be addressed. Using analytic techniques grounded in differential subordination, subordination chains, and classical coefficient inequalities, we derived explicit constraints for the initial coefficients and bounds of Fekete-Szgo.

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