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تقدير معلمات التوزيع اللوجستي الأسي: نهج طريقة MLE مقابل طريقة MPS

"Estimating Logistic-Exponential Parameters: MLE ver MPS Approach"

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Keywords

The Logistic-Exponential distribution, Maximum Likelihood Estimation, Maximum Product of Spacings, Simulation Study, Failure Time.

Abstract

"This article compares the Maximum Likelihood Estimation (MLE) and Maximum Product of Spacings (MPS) methods for estimating the parameters of the Logistic-Exponential (LE) distribution. A two-stage approach was followed. In the first stage, a simulation study, which considered three scenarios for each shape (kappa) and scale (lambda) parameters, with sample sizes of 25, 50 and 100, was carried out. The results of the simulations showed that as the sample size increased, both methods were consistent, and the MPS method outperformed the MLE method in terms of Mean Squared Error (MSE), especially in small samples. In the second stage, a real lifetime data set was fitted using the LE distribution. The results demonstrated that MPS outperformed the MLE method in terms of MSE (0.00127 against 0.00175 for MLE). The LE distribution is very flexible for modeling failure time data. The results obtained here show that MPS is a more efficient method than MLE for estimating the parameters when the available data are limited.

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التوزيع اللوجستي-الأسي، تقدير الإمكان الاعظم، حاصل ضرب المسافات الأقصى، دراسة محاكاة، زمن الفشل."

ملخص

تقارن هذه المقالة بين طريقتي تقدير الإمكان الأعظم (MLE) وأقصى حاصل ضرب للمسافات (MPS) لتقدير معلمات التوزيع اللوجستي-الأسي (LE). حيث تم اتباع منهجية من مرحلتين. في المرحلة الأولى، أُجريت دراسة محاكاة، شملت ثلاثة سيناريوهات لكل من معلمات الشكل (κ) والمقياس (λ)، بأحجام عينات ٢٥ و٥٠ و١٠٠. أظهرت نتائج المحاكاة أتساق الطريقتين مع زيادة حجم العينة، وتفوق طريقة MPS على طريقة MLE من حيث متوسط مربع الخطأ (MSE)، خصوصاً في العينات الصغيرة. وفي المرحلة الثانية، دراسة مجموعة بيانات حقيقية لعمر الخدمة باستخدام توزيع LE. وأظهرت النتائج أن طريقة MPS تفوقت على طريقة MLE من حيث MSE (٠.٠٠١٢٧ مقابل ٠.٠٠١٧٥ لطريقة MLE). يتميز توزيع LE بمرونة عالية في نمذجة بيانات وقت الفشل. وتُظهر النتائج التي تم الحصول عليها هنا أن طريقة MPS أكثر كفاءة من طريقة MLE عند محدودية البيانات.

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1. Introduction

Probability distributions are the backbone of statistical modeling and reliability. These are applied to solve various problems in medical, engineering, and environmental fields. Apart from many generalized forms, some distributions have been introduced to improve on exponential distribution which is a basic baseline (Ali et al., 2025; Saboor et al., 2025). It has constant hazard rate, and researchers have been striving to propose models that have varying hazard rates. This new generation of distributions provides a flexible class of lifetime models for various applications (Okasha & Nassar, 2022). Among the added members of this family, the Logistic-Exponential (LE) distribution is a flexible new generation distribution introduced by "Lan and Leemis (2008)" to get the advantage of two renowned distributions simultaneously. It combines the characteristics of logistic distribution and exponential distribution (Dutta & Kayal, 2024; Mansoor et al., 2020).

This probability distribution is called the LE distribution. It has a two-parameters probability density function (pdf) with shape parameter (κ) and scale parameter (λ). The LE distribution can represent increasing failure rate (IFR) and decreasing failure rate (DFR) populations, as well as populations in the shape of a bathtub (BT) or upside-down bathtub (UBT) (Lan & Leemis, 2008). The LE distribution was named as a 'Logistic-Exponential' distribution because its survival function is similar to that of log-logistic distribution. The LE distribution has gamma-like tails as $t \to \infty$, but behave as log-logistic distributions close to zero (Lan & Leemis, 2008). Therefore, the LE distribution is suitable for analyzing lifetime data with complex behaviors that cannot be addressed by existing distributions, such as Weibull distribution and Exponential distribution (Jeon et al., 2020; Alsaggaf, 2024).

The performance of the proposed statistical model largely depends on how well its parameters are estimated in real situations. Maximum Likelihood Estimation (MLE) is the commonly used frequentist technique for parameter estimation which enjoys desirable asymptotic properties. However, establishing its applicability in real-life situations is another issue; several studies have addressed related issues (Sadiq et al., 2023). For example, MLE is known to suffer from severe practical difficulties in acquiring consistent and well-defined estimates for small samples, J-shaped stable distributions, and unbounded likelihoods (Anatolyev & Kosenok, 2005; Smith, 1985). Furthermore, estimation of the shape parameter in LE-type models is notoriously sensitive, potentially leading to instability if the parameters reside near the boundaries of the parameter space (Cheng & Amin, 1983; Akram & Hayat, 2014).

In order to overcome the shortcomings of the methodological restrictions faced by MLE, the use of Maximum Product of Spacings (MPS) method has been proposed as a competitive alternative. The method, first introduced by Cheng and Amin in (1983) and then by Ranneby (1984), was based on maximizing the geometric mean of probability spacings based on the underlying CDF. The method does an excellent job of assigning a uniform distribution, which achieves excellent accuracy and performs better than MLE methods, especially in cases of limited samples or non-regular cases where MLE fails (Alotaibi et al., 2025; Nassar et al., 2023). It is well known in the literature that MPS estimators achieve all the strengths of large samples (such as consistency and efficiency) and are actually superior in terms of good fit to the distribution at hand, especially at the tails (Shao & Hahn, 1999); and (Okasha & Nassar, 2022).

Despite the advantages provided by the Logistic-Exponential (LE) distribution and other methods for estimating its parameters, few studies have made a comparative analysis between using the (MLE) and (MPS) for determining the parameters of LE distribution under different simulation conditions (Sabry et al., 2022). The current study fills this scholarly void by conducting a comparative efficiency study between the two estimators of the LE distribution. The study explores the efficiency of the studied estimators and analyzes their behavior using different simulation models. A detailed simulation study was performed using Python software for 1000 sets of simulated data from different sample sizes. For each iteration, the data were generated using the Inverse Transform Sampling method by using the quantile function of the studied distribution. The findings were then applied to real life failure time data, in order to test the ability of MPS to achieve minimum Mean Squared Error (MSE) and to obtain a more realistic behavior of survival function and hazard function than MLE, which makes it an efficient method to be used as a guide for reliability study of small sample size.

2. The Logistic-Exponential Distribution: Properties and Mathematical Functions

Logistic-Exponential (LE) distribution is a continuous probability model that is very flexible and recently introduced as a generalization to exponential distribution to accommodate well to complex survival time data (Lan & Leemis, 2008). This new generalization offers the advantages of simple exponential distribution along with the flexibility of logistic distribution. So it can capture very general shapes of failure time data (Ali et al., 2020).

2.1 Probability Density Function (PDF)

For a random variable T following Logistic-Exponential distribution with shape parameter κ and scale parameter λ the probability density function has the form (Lan & Leemis, 2008):

$$f(t;\kappa,\lambda)=\frac{\kappa\lambda e^{\lambda t}\left(e^{\lambda t}-1\right)^{\kappa-1}}{[1+(e^{\lambda t}-1)^{\kappa}]^2},\quad t>0,\kappa,\lambda>0$$

This function demonstrates a high capacity to take both symmetric and asymmetric shapes depending on the parameter values, making it capable of simulating varying degrees of skewness and kurtosis (Alsaggaf, 2024).

2.2 Cumulative Distribution Function (CDF)

The cumulative distribution function of the LE distribution is defined by its closed mathematical form, which is a key advantage that facilitates estimation processes, particularly in the Maximum Product of Spacings (MPS) method (Nassar et al., 2023). It is formulated as follows (Lan & Leemis, 2008):

$$F(t;\kappa,\lambda)=\frac{\left(e^{\lambda t}-1\right)^{\kappa}}{1+(e^{\lambda t}-1)^{\kappa}}$$

It is noteworthy that when the shape parameter $\kappa=1$, the distribution reduces exactly to the traditional exponential distribution (Ali et al., 2020).

2.3 Survival and Hazard Functions

The survival function $S(t)$ and the hazard function $h(t)$ are among the most important tools in reliability analysis. The survival function for this distribution is (Lan & Leemis, 2008):

$$S(t;\kappa,\lambda)=\frac{1}{1+(e^{\lambda t}-1)^{\kappa}}$$

As for the hazard function, it is the most prominent feature of the LE distribution, as it possesses the unique ability to represent five different failure patterns: constant, increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped (Dutta & Kayal, 2024). It is mathematically defined as (Lan & Leemis, 2008):

$$h(t;\kappa,\lambda)=\frac{\kappa\lambda e^{\lambda t}\left(e^{\lambda t}-1\right)^{\kappa-1}}{1+(e^{\lambda t}-1)^{\kappa}}$$

2.4 Quantile Function

For simulation and random data generation purposes, the quantile function t_p is used, which is obtained through the inverse of the cumulative distribution function (Lan & Leemis, 2008):

$$t_p=\frac{1}{\lambda}\ln\left[1+\left(\frac{p}{1-p}\right)^{1/\kappa}\right],\quad 0<p<1$$

This function serves as the mathematical foundation for the Inverse Transform Sampling algorithm used in this research to generate simulation samples, ensuring an accurate transformation of uniform variables into variables that follow the LE distribution (Sadiq et al., 2023).

.3 Estimation Methods

In statistical modeling, it becomes very important to choose a good estimator that possesses some desirable properties of a good estimator such as consistency and efficiency (Sadiq et al., 2023). In this section, we discuss and compare two methods to compare and estimate the parameters of LE distribution.

1. Maximum Likelihood Estimators (MLE)

Let $t_1, t_2, \dots, t_n$ be a random sample of independent and identically distributed $(i.i.d.)$ observations drawn from a Logistic-Exponential (LE) distribution with parameters (κ, λ). The probability density function (PDF) is given by (Lan & Leemis, 2008):

$$f(t;\kappa,\lambda) = \frac{\kappa\lambda e^{\lambda t}(e^{\lambda t}-1)^{\kappa-1}}{[1+(e^{\lambda t}-1)^{\kappa}]^2}, t > 0, \kappa, \lambda > 0$$

The Likelihood Function (Lan & Leemis, 2008):

$$L(\kappa,\lambda|\underline{t}) = \prod_{i=1}^{n} f(t_i;\kappa,\lambda) = (\kappa\lambda)^n \prod_{i=1}^{n} \frac{e^{\lambda t_i}(e^{\lambda t_i}-1)^{\kappa-1}}{[1+(e^{\lambda t_i}-1)^{\kappa}]^2}$$

The Log-Likelihood Function (Lan & Leemis, 2008):

By taking the natural logarithm of both sides ($\ell = \ln L$), we obtain:

$$\ell = n\ln\kappa + n\ln\lambda + \lambda\sum_{i=1}^{n} t_i + (\kappa - 1)\sum_{i=1}^{n} \ln(e^{\lambda t_i}-1) - 2\sum_{i=1}^{n} \ln[1+(e^{\lambda t_i} - 1)^{\kappa}]$$

Likelihood Equations:
To determine the maximum likelihood estimates, we take the partial derivatives of ℓ with respect to κ and λ and equate them to zero (Lan & Leemis, 2008):

- Partial derivative with respect to κ:

$$\frac{\partial \ell}{\partial \kappa} = \frac{n}{\kappa} + \sum_{i=1}^{n} \ln(e^{\lambda t_i}-1) - 2\sum_{i=1}^{n} \frac{(e^{\lambda t_i}-1)^{\kappa}\ln(e^{\lambda t_i}-1)}{1+(e^{\lambda t_i}-1)^{\kappa}} = 0$$

- Partial derivative with respect to λ:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} t_i + (\kappa-1)\sum_{i=1}^{n} \frac{t_i e^{\lambda t_i}}{e^{\lambda t_i}-1} - 2\sum_{i=1}^{n} \frac{\kappa t_i e^{\lambda t_i}(e^{\lambda t_i}-1)^{\kappa-1}}{1+(e^{\lambda t_i}-1)^{\kappa}} = 0$$

Since these equations are non-linear and complex, they cannot be solved analytically for closed-form expressions of $\hat{\kappa}$ and $\hat{\lambda}$ (Lan & Leemis, 2008). Therefore, numerical methods, such as the **Newton-Raphson algorithm**, are employed to obtain the estimates (Almetwally & Almongy, 2019; Nassar & Elshahhat, 2023).

2. Maximum Product of Spacings (MPS) Estimators

The MPS method is based on the principle of distributing probabilities uniformly among the intervals (spacings) of the cumulative distribution function (CDF) (Cheng & Amin, 1983; Ranneby, 1984). The CDF for the LE distribution is defined as (Lan & Leemis, 2008):

$$F(t;\kappa,\lambda)=\frac{(e^{\lambda t}-1)^{\kappa}}{1+(e^{\lambda t}-1)^{\kappa}}$$

Assuming an ordered sample $t_{(1)}<t_{(2)}<\cdots<t_{(n)}$, we define the uniform spacings D_i as follows:

- Initial spacing: $D_1=F(t_{(1)};\kappa,\lambda)$
- Internal spacings: $D_i=F(t_{(i)};\kappa,\lambda)-F(t_{(i-1)};\kappa,\lambda)\,, i=2,\dots,n$
- Terminal spacing: $D_{n+1}=1-F(t_{(n)};\kappa,\lambda)$

Where $\sum_{i=1}^{n+1} D_i=1$. The product of spacings function is defined as (Cheng & Amin, 1983):

$$G(\kappa,\lambda)=\left[\prod_{i=1}^{n+1} D_i\right]^{\frac{1}{n+1}}$$

Log-Product Spacing Function:

$$H(\kappa,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i$$

Substitution of the Spacings Formula: By substituting the CDF $F(t)$ into $H(\kappa,\lambda)$, we yield:

$$\begin{aligned}H(\kappa,\lambda)&\\=\frac{1}{n+1}&\left[\ln\left(\frac{(e^{\lambda t_{(1)}}-1)^{\kappa}}{1+(e^{\lambda t_{(1)}}-1)^{\kappa}}\right)\right.\\&+\sum_{i=2}^{n}\ln\left(\frac{(e^{\lambda t_{(i)}}-1)^{\kappa}}{1+(e^{\lambda t_{(i)}}-1)^{\kappa}}\right.\\&\left.-\frac{(e^{\lambda t_{(i-1)}}-1)^{\kappa}}{1+(e^{\lambda t_{(i-1)}}-1)^{\kappa}}\right)\\&\left.+\ln\left(1-\frac{(e^{\lambda t_{(n)}}-1)^{\kappa}}{1+(e^{\lambda t_{(n)}}-1)^{\kappa}}\right)\right]\end{aligned}$$

The MPS estimators are obtained by solving the system of equations resulting from setting the partial derivatives of H to zero ($\frac{\partial H}{\partial\kappa}=0$ and $\frac{\partial H}{\partial\lambda}=0$). Research confirms that MPS is particularly effective for skewed models like the LE distribution (Nassar et al., 2023; Hassan et al., 2025).

٤. Simulation Study

To evaluate the performance and efficiency of Maximum Likelihood (MLE) and Maximum Product of Spacing (MPS) estimators for parameters in a Logistic-Exponential distribution, an extensive simulation study was conducted using the Python programming language. This study aimed to compare the two methods under different experimental conditions to demonstrate the consistency and accuracy of the estimators.

٤.١ Simulation Design

The simulation study was designed according to the following methodological steps:

1. Data Generation: Random samples were generated from the Logistic-Exponential (LE) distribution using the Inverse Transform Sampling Algorithm, based on the previously defined Quantile Function of the distribution.
2. Sample Sizes: Three different sample sizes were selected to represent small, medium, and large cases: $(n = 25, 50, 100)$.
3. Scenarios (Parameter Values): Three scenarios for the parameters (κ, λ) were tested to cover various shapes of the hazard function:

- Scenario I: $\kappa = 0.5$ (where the shape parameter is less than one) with λ values of $(٠.٠٥, ٠.١٠, ٠.١٥)$.
- Scenario II: $\kappa = 1$ (the case where the distribution reduces to the Exponential distribution) with λ values of $(٠.٠٥, ٠.١٠, ٠.١٥)$.
- Scenario III: $\kappa = 1.5$ (where the shape parameter is greater than one) with λ values of $(٠.٠٥, ٠.١٠, ٠.١٥)$.

4. Replications: The simulation was repeated 1,000 times for each case to ensure the stability of the results and to minimize random error.
5. Comparison Criterion: The Mean Squared Error (MSE) was utilized as the primary criterion to compare the accuracy and performance of the two estimators.

٤.٢ Presentation and Analysis of Simulation Results

Tables (1) through (5) encapsulate the numerical findings of the simulated experimental scenarios. These results are systematically categorized based on structural transitions in the hazard function's behavior and their subsequent impact on estimation precision, detailed as follows:

Scenario I - Table (1): This scenario investigates estimation performance when the shape parameter is less than unity $(\kappa < 1)$. Statistically, this configuration represents a Decreasing Failure Rate (DFR) model.

Scenario II - Table (2): This case explores the standard baseline where the shape parameter converges to unity $(\kappa = 1)$. Mathematically, the LE model reduces to the classical

Exponential distribution characterized by a constant hazard rate.

Scenario III - Table (3): This scenario examines the transitional phase towards an Increasing Hazard Rate (IHR). The focus here is on monitoring estimator efficiency as the probability density function (PDF) begins to exhibit rightward skewness.

Scenario IV - Table (4): This section models statistical behavior under conditions of escalating distribution intensity and rapid hazard rate growth.

Scenario V - Table (5): This scenario represents the extreme case of shape parameter growth (High Shape Parameter).

These scenarios jointly provide a stress test environment to study the asymptotic properties as well as small sample efficiency of the developed estimators for their validity and accuracy over the entire range of hazard functions of the LE distribution.

Table (1): Results of the simulation of: $\kappa = 0.5,\ \lambda = 0.05, 0.10, 0.15$

n	**Parameter**	**True Val**	**MPS MSE**	**MLE MSE**
25	Kappa	0.50	0.012255	0.015551
25	Lambda	0.05	0.000377	0.000388
50	Kappa	0.50	0.006063	0.006682
50	Lambda	0.05	0.000145	0.000149
100	Kappa	0.50	0.002758	0.002857
100	Lambda	0.05	0.000067	0.000069
25	Kappa	0.50	0.012647	0.015802
25	Lambda	0.10	0.001437	0.001439
50	Kappa	0.50	0.005455	0.006156
50	Lambda	0.10	0.000605	0.000635
100	Kappa	0.50	0.002798	0.002792
100	Lambda	0.10	0.000272	0.000276
25	Kappa	0.50	0.011888	0.015242
25	Lambda	0.15	0.003078	0.003113
50	Kappa	0.50	0.005651	0.005938
50	Lambda	0.15	0.001289	0.001369
100	Kappa	0.50	0.002557	0.002626
100	Lambda	0.15	0.000586	0.000599

Table (2): Results of the simulation of: $\kappa = 1,\ \lambda = 0.05, 0.10, 0.15$

n	**Parameter**	**True Val**	**MPS MSE**	**MLE MSE**
25	Kappa	1	0.036411	0.045314
25	Lambda	0.05	0.000136	0.000140
50	Kappa	1	0.017462	0.019702
50	Lambda	0.05	0.000062	0.000063
100	Kappa	1	0.008102	0.008720

100	Lambda	0.05	0.000031	0.000032
25	Kappa	1	0.035504	0.046740
25	Lambda	0.10	0.000529	0.000545
50	Kappa	1	0.018215	0.018687
50	Lambda	0.10	0.000274	0.000281
100	Kappa	1	0.008749	0.008835
100	Lambda	0.10	0.000131	0.000133
25	Kappa	1	0.036539	0.044842
25	Lambda	0.15	0.001293	0.001310
50	Kappa	1	0.016674	0.017498
50	Lambda	0.15	0.000521	0.000531
100	Kappa	1	0.008953	0.009153
100	Lambda	0.15	0.000266	0.000270

Table (3): Results of the simulation of: $\kappa = 1.5,\ \lambda = 0.05, 0.10, 0.15$

n	**Parameter**	**True Val**	**MPS MSE**	**MLE MSE**
25	Kappa	1.5	0.075096	0.086695
25	Lambda	0.05	0.000072	0.000073
50	Kappa	1.5	0.034358	0.036046
50	Lambda	0.05	0.000031	0.000031
100	Kappa	1.5	0.018030	0.018420
100	Lambda	0.05	0.000015	0.000015
25	Kappa	1.5	0.079906	0.097822
25	Lambda	0.10	0.000271	0.000271
50	Kappa	1.5	0.034815	0.039322
50	Lambda	0.10	0.000132	0.000134
100	Kappa	1.5	0.019044	0.019570
100	Lambda	0.10	0.000063	0.000064
25	Kappa	1.5	0.074032	0.090394
25	Lambda	0.15	0.000574	0.000580
50	Kappa	1.5	0.034271	0.036504
50	Lambda	0.15	0.000272	0.000275
100	Kappa	1.5	0.017939	0.017730
100	Lambda	0.15	0.000149	0.000150

Table (4): Results of the simulation of: $\kappa = 2,\ \lambda = 0.05, 0.10, 0.15$

n	**Parameter**	**True_Val**	**MPS_MSE**	**MLE_MSE**
25	Kappa	2	0.127234	0.153091
25	Lambda	0.05	0.000040	0.000040
50	Kappa	2	0.069150	0.070154
50	Lambda	0.05	0.000019	0.000019
100	Kappa	2	0.032516	0.035398
100	Lambda	0.05	0.000008	0.000008
25	Kappa	2	0.135833	0.171998
25	Lambda	0.10	0.000156	0.000156
50	Kappa	2	0.060939	0.062543

50	Lambda	0.10	0.000076	0.000077
100	Kappa	2	0.032151	0.032311
100	Lambda	0.10	0.000038	0.000038
25	Kappa	2	0.130851	0.160962
25	Lambda	0.15	0.0003٦٩	0.000369
50	Kappa	2	0.065625	0.068138
50	Lambda	0.15	0.000172	0.000173
100	Kappa	2	0.033105	0.033532
100	Lambda	0.15	0.000084	0.000084

Table (5): Results of the simulation of: $\kappa = 2.5,\ \lambda = 0.05, 0.10, 0.15$

n	Parameter	True_Val	MPS_MSE	MLE_MSE
25	Kappa	2.50	0.193036	0.208437
25	Lambda	0.05	0.000025	0.000025
50	Kappa	2.50	0.098233	0.100239
50	Lambda	0.05	0.000011	0.000011
100	Kappa	2.50	0.048208	0.049563
100	Lambda	0.05	0.000006	0.000006
25	Kappa	2.50	0.208880	0.265018
25	Lambda	0.10	0.000093	0.000093
50	Kappa	2.50	0.098743	0.103302
50	Lambda	0.10	0.000050	0.000050
100	Kappa	2.50	0.045824	0.047927
١00	Lambda	0.10	0.000024	0.000025
25	Kappa	2.50	0.205394	0.248685
25	Lambda	0.15	0.000222	0.000222
50	Kappa	2.50	0.094019	0.097586
50	Lambda	0.15	0.000115	0.000115
100	Kappa	2.50	0.051815	0.050149
100	Lambda	0.15	0.000053	0.000053

The tables for the outcomes (Tables 1 through 5) of the simulation study present comparative results for mean squared error of estimators of parameters of the Logistic-Exponential (LE) distribution obtained from two estimators: Maximum Likelihood Estimators (MLE) and Maximum Product of Spacings (MPS).:

Across all simulated scenarios, both the MLE and MPS methods demonstrate the fundamental property of consistency. Specifically, a monotonic decrease in the Mean Squared Error (MSE) values for both parameters (κ,λ) is observed as the sample size increases from 25 to 100. This convergence towards the true parameter values signifies that both estimators effectively utilize the increasing information provided by larger samples, a finding that aligns with the theoretical expectations and empirical results documented by Ekström et al. (2019) and Ali et al. (2025).

In terms of methodological superiority, the results highlight a distinct advantage for the Maximum Product of Spacings (MPS) method in estimating the shape parameter (κ), particularly in small-sample contexts ($n = 25$). For instance, in the third scenario (Table 3), with true values of $\kappa = 1.5\ and\ \lambda = 0.10$, the MPS method yielded an MSE of 0.079906 compared to 0.097822 for the MLE. This superior performance in restricted samples reflects the inherent robustness of the MPS philosophy, which excels in handling skewed distributions and tail complexities where the likelihood function may become unstable or fail to provide a unique maximum observations that are consistent with the findings of Anatolyev and Kosenok (2005) and Alslman and Helu (2022).

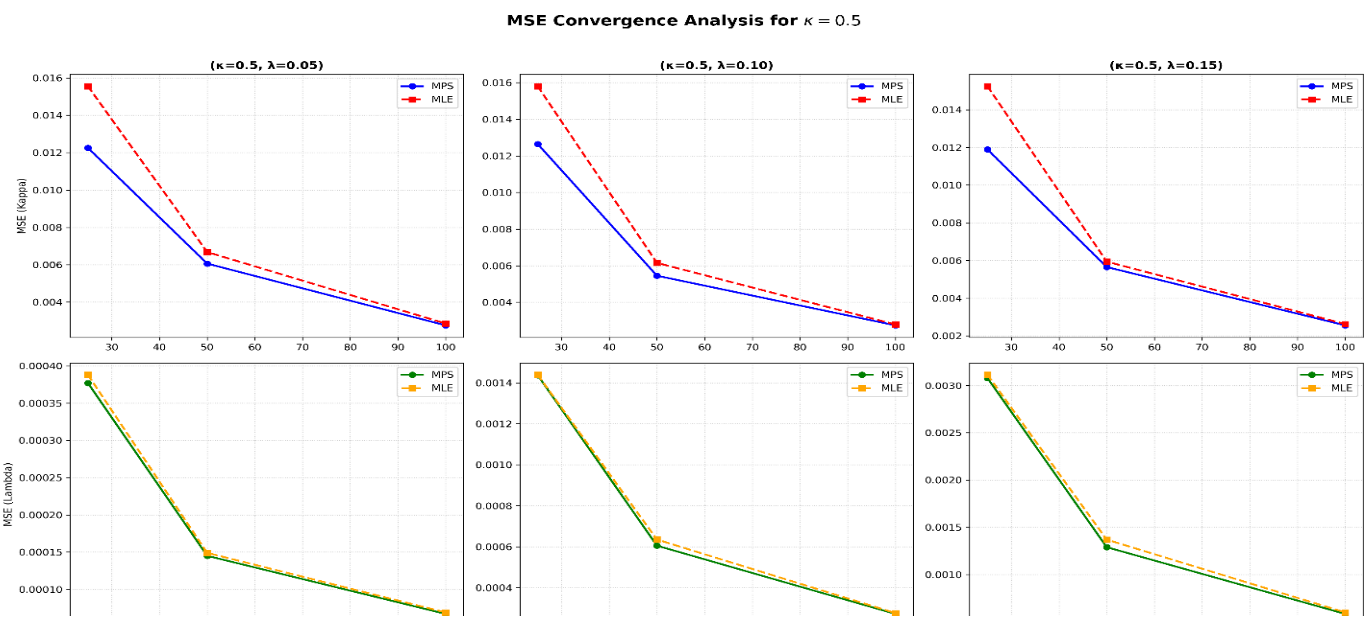
Regarding the scale parameter (λ), both estimation techniques exhibit a high degree of precision and convergence. While the MPS method holds a marginal edge in very small samples, the performance gap between the two methods effectively vanishes as the sample size reaches $n = 100$, with MSE values dropping to the magnitude of $10^{-5} and 10^{-6}$. This high level of stability confirms the LE distribution's efficiency in scale parameter estimation regardless of the frequentist approach used, provided sufficient data is available, mirroring the asymptotic equivalence noted by Cheng and Amin (1983) and Ranneby (1984).

Furthermore, the simulation reveals that the estimation of the shape parameter (κ) becomes increasingly sensitive as its true value rises. The MSE for κ grew from approximately 0.012 in the first scenario ($\kappa = 0.5$) to 0.193 in the fifth scenario ($\kappa = 2.5$). This proportional increase in error indicates that the statistical model becomes more susceptible to parameter fluctuations when the hazard rate exhibits a strong increasing trend. Such sensitivity necessitates the use of more robust estimators like MPS to mitigate potential bias, as suggested by the research of Almetwally and Almongy (2019).

In conclusion, these empirical results demonstrate that while both methods are statistically valid, the MPS method serves as the most reliable and precise choice for researchers dealing with limited sample sizes or when the primary focus is the accurate estimation of the shape parameter, consistently outperforming the traditional MLE in minimizing mean squared errors in such environments.

Figures 1–5 illustrate a considerable amount of agreement between the figures and the Tables 1–5. Monotonic decline of the figures is consistent with statistical consistency proved numerically. For all considered scenarios, with the increase in sample size (n) both considered estimators are proven to be consistent and shown to approach zero limit of Mean Squared Error (MSE).

Figure (1): Diagram of the results of: $\kappa = 0.5$, $\lambda = 0.05, 0.10, 0.15$



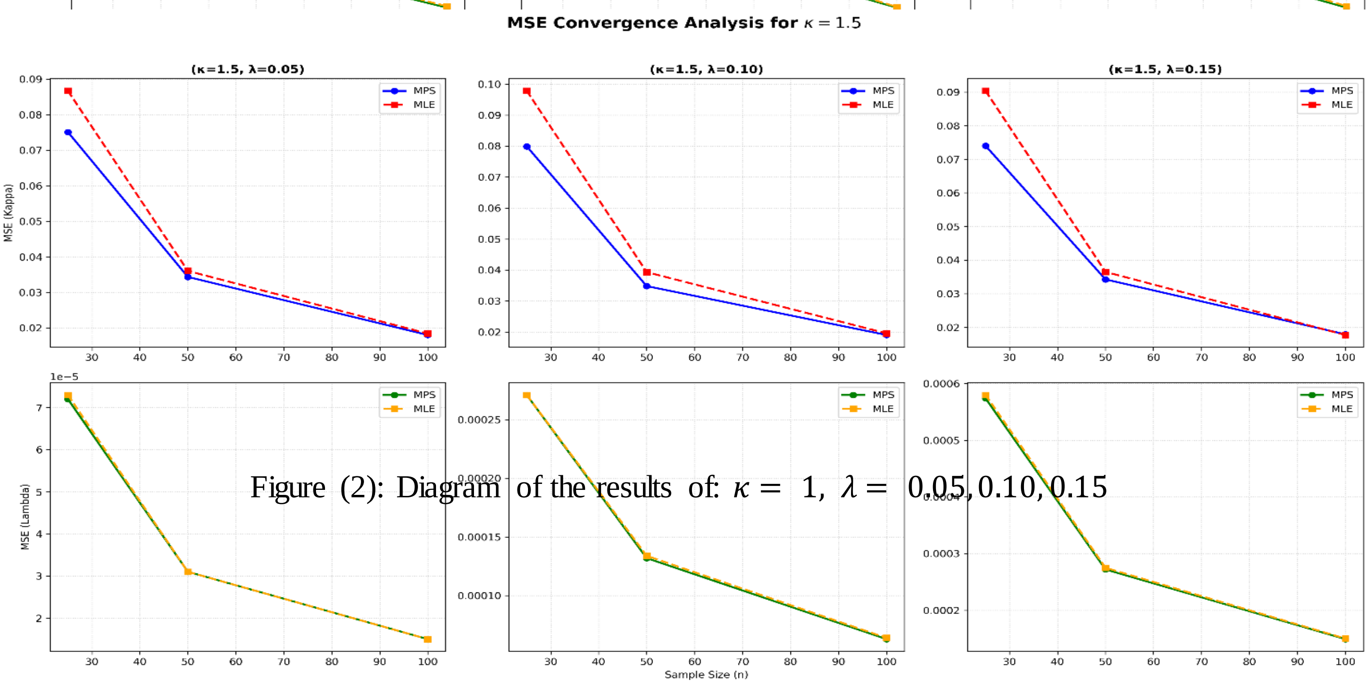


Figure (2): Diagram of the results of: $\kappa = 1$, $\lambda = 0.05, 0.10, 0.15$

Figure (3): Diagram of the results of: $\kappa = 1.5,\ \lambda = 0.05, 0.10, 0.15$

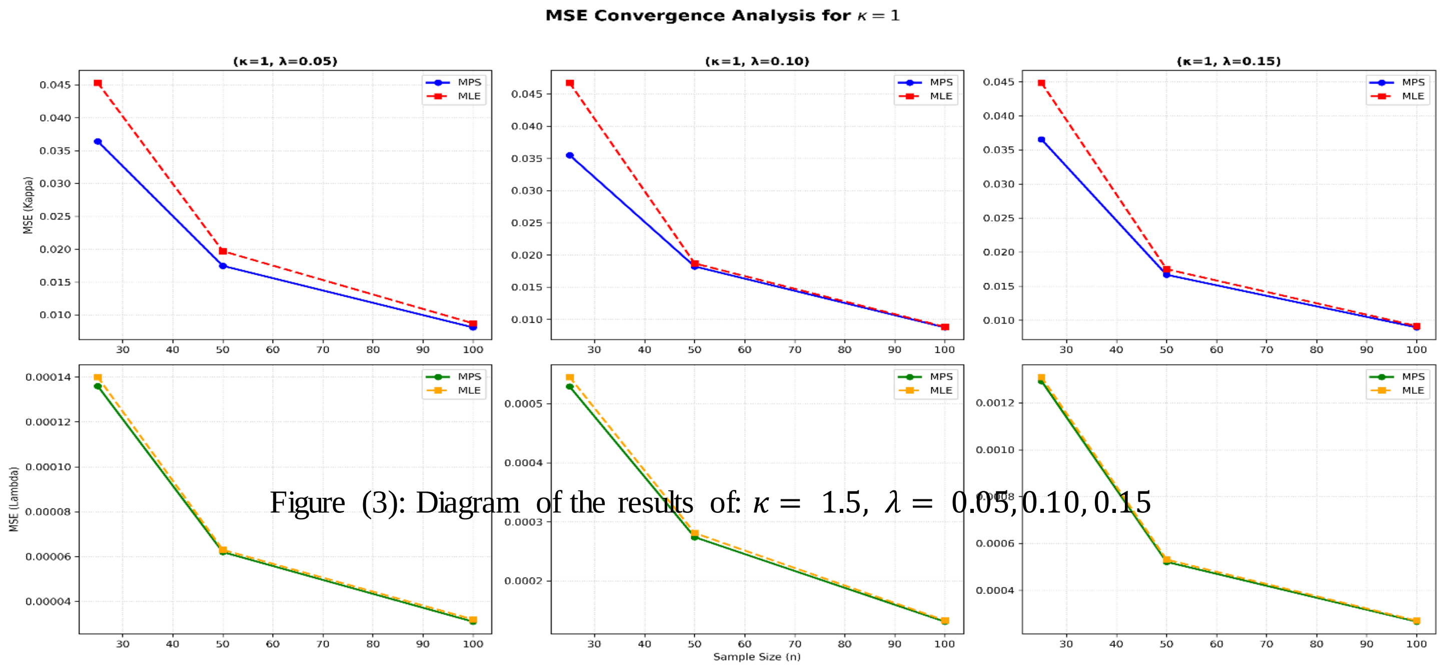


Figure (4): Diagram of the results of: $\kappa = 2,\ \lambda = 0.05, 0.10, 0.15$

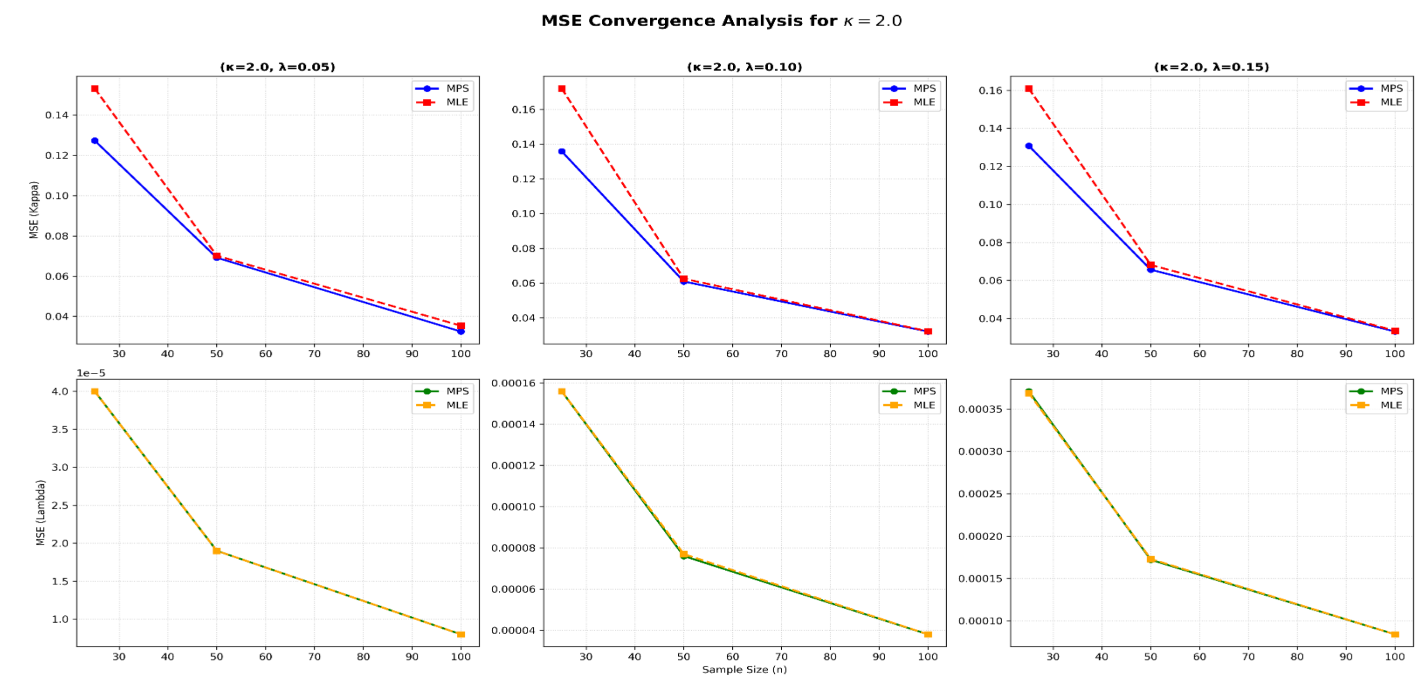
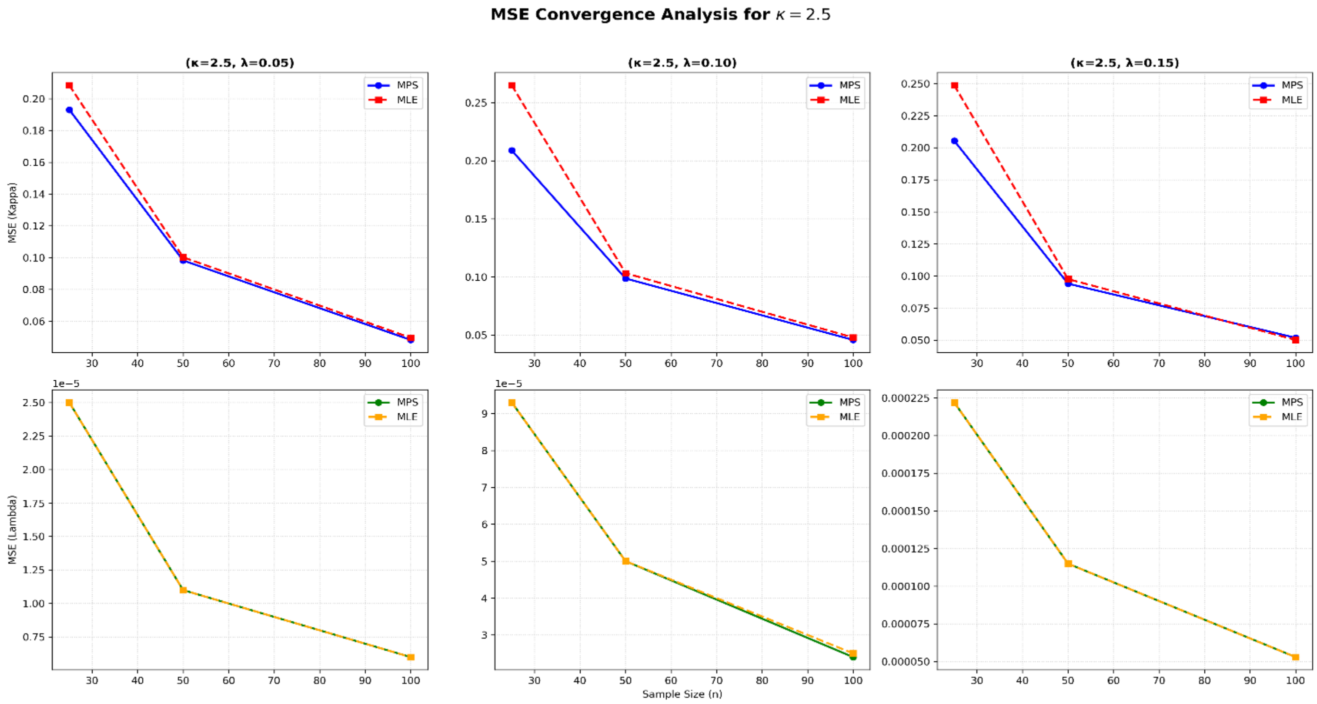


Figure (5): Diagram of the results of: $\kappa = 2.5,\ \lambda = 0.05, 0.10, 0.15$



5. Application to Real Data: Performance Validation

5.1 Data Source and Description

To evaluate the practical applicability of the Logistic-Exponential (LE) distribution and to compare the performance of MLE and MPS estimators, we utilized a well-known real-world dataset from the field of reliability engineering. The data were obtained from the seminal study by Lan and Leemis (2008).

The dataset, referred to as the "first data set" in the source, consists of complete survival data representing the failure times (in hours) of a specific type of components. have been widely used in statistical literature to test the flexibility of new lifetime distributions.

The dataset is as follows (n = 23):

{17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40}

Rationale for Selection

This dataset was selected because it is a complete sample (no censoring), which provides a rigorous environment to test the convergence and efficiency of the Maximum Product of Spacings (MPS) versus the Maximum Likelihood Estimation (MLE). According to Lan and Leemis

(2008), this data set exhibits an increasing failure rate (IFR), making it an ideal candidate for the LE distribution's shape parameters where $\kappa > 1$.

5.2 Goodness-of-Fit Analysis

To verify the suitability of the Logistic-Exponential (LE) distribution for the real-dataset, the Chi-square (χ^2) goodness-of-fit test was conducted. This test measures the statistical compatibility between the proposed probability distribution and the observed data, a methodology widely adopted for evaluating reliability models and their practical applications (Samih & Alwan, 2024).

The test statistic was calculated with one degree of freedom ($df = 1$) at a significance level of 0.05, where the critical tabulated value is 3.841. Table (4) presents the results of the goodness-of-fit tests, including the χ^2 values, P-values, and the Anderson-Darling (A-D) statistic for both estimation methods.

Table (6): Chi-square Goodness-of-Fit Test Results

الطريقة	χ^2	df	P-value (Chi2)	Anderson-Darling
MLE	2.5251	1	0.47077	0.21905
MPS	2.0075	1	0.57085	0.26532

Discussion and Decision:

Upon examining the results, it is observed that the calculated χ^2 values for both methods (2.5251 for MLE and 2.0075 for MPS) are lower than the critical tabulated value of 3.841. Additionally, the P-values for both methods are greater than the significance level ($\alpha = 0.05$), leading to the failure to reject the null hypothesis (H_0). This confirms the high efficiency of the Logistic-Exponential distribution in representing the real survival data.

Furthermore, the MPS method demonstrates a clear superiority by yielding the lowest test statistic, indicating a superior alignment between the model and the observed data compared to the Maximum Likelihood approach. This confirms that the spacing-based estimation provides a more refined fit for the component failure times originally discussed by Lan and Leemis (2008).

٥.٣ Empirical Validation and Comparative Assessment

Analysis results show that fitting the Logistic-Exponential (LE) distribution to failure-time data and using Mean Squared Error (MSE) to evaluate methods, the following analytical insights were derived:

- General Performance and Structural Fit: Both the MLE and MPS procedures demonstrated good performance in terms of fitting the empirical data. Figure (4) illustrates goodness of fit across the four fundamental functions for both estimators, with the theoretical distributions generated by the two estimators, showing good fit to the actual data distribution.
- The Precision of the MPS Paradigm: The Maximum Product of Spacings (MPS) method achieves a standard of fitting precision with an MSE of 0.00127. This performance confirms the framework's efficacy in minimizing the divergence between theoretical postulates and experimental observation.
- Precision of the MLE Paradigm: The Maximum Likelihood Estimation (MLE) method yielded an MSE of 0.00175. While this value remains impressively low and reflects excellent model adherence, it was marginally less precise than the MPS alternative for this specific reliability application.

Graphical Analysis

Figure (6) illustrates the functional forms of $f(t), F(t), S(t), and\ h(t)$ for the empirical dataset, revealing several critical behaviors:

1. **Probability Density $\boldsymbol{f(t)}$**: Graphical evidence indicates that the MPS curve ($\hat{\kappa} \approx 2.018$) provided a more equilibrium-based representation of the data's primary peak compared to the MLE trajectory ($\hat{\kappa} \approx 2.388$), which appeared slightly more constrained.
2. **Cumulative Distribution $\boldsymbol{F(t)}$**: The trajectory of the following the MPS algorithm more closely matches the empirical "stepladder" CDF. This closer matching of the analytical model to can be seen to offer a justification of sorts

for the lower MSE found with the MPS algorithm.

3. **Survival Function $S(t)$:** The survival plot depicts a logically consistent and progressive decay in survival probability over the temporal domain. Notably, the MPS approach exhibited superior structural plasticity and precision in the distribution's tail (high-duration values), suggesting enhanced predictive reliability for long-lived components compared to the MLE framework.
4. **Hazard Rate $h(t)$:** The analysis of the LE-station reveals that the observed LE-distribution characterizes an Increasing Hazard Rate (IHR). This property supports the choice of the Logistic-Exponential model for the failure-time data.

The findings derived from the real application scenario concur with those obtained from the multi-scenario simulation study. Specifically, the findings from both study components support the use of MPS method for being superior than other methodological alternatives under restricted sample sizes and/or observational data. Thus, as a major methodological contribution to the area, the findings of this study recommend the use of the Maximum Product of Spacings (MPS) method as the first preferred method for estimating the parameters of the Logistic-Exponential distribution to maximize the goodness-of-fit and to minimize any potential predictive error.

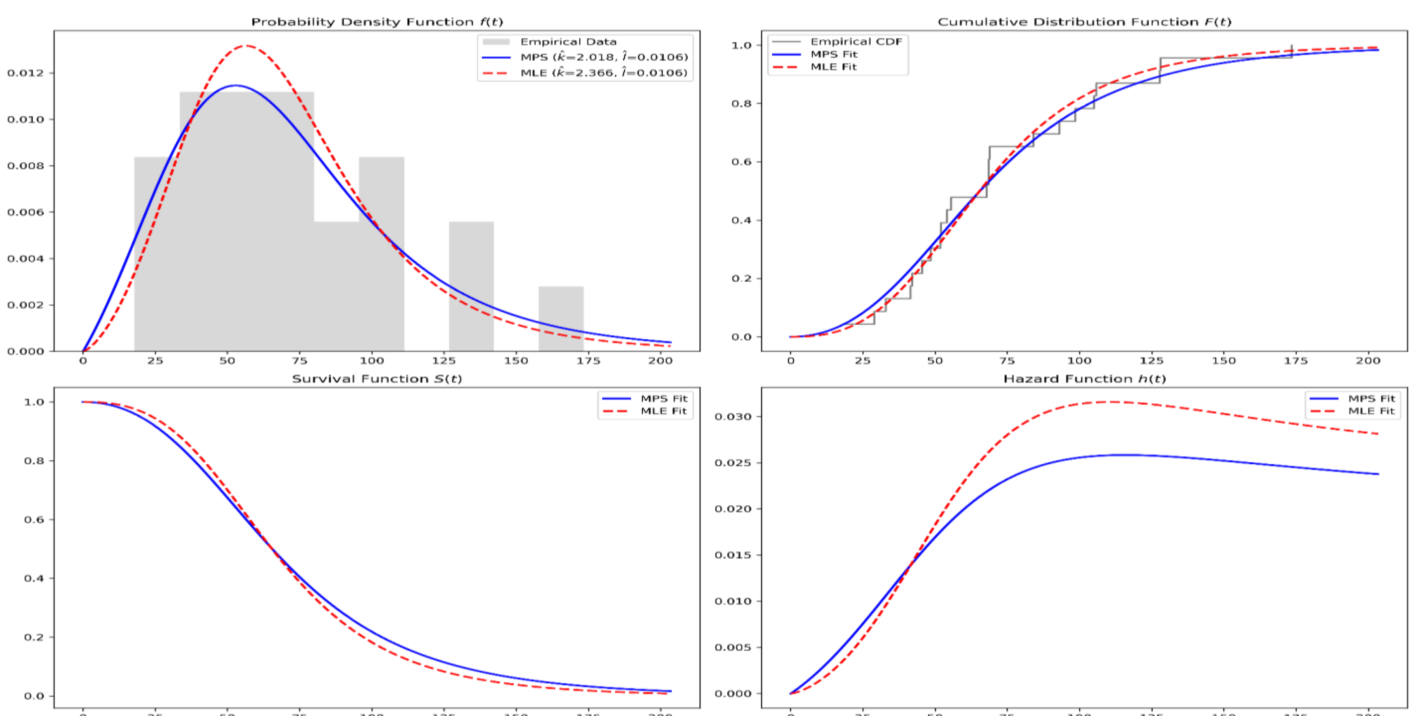


Figure (6): Shows the functions $f(t), F(t), S(t), h(t)$ for real data.

5. Results and Discussion

This section provides a rigorous synthesis of the simulation experimental outcomes and the subsequent empirical validation, specifically contrasting the estimation performance of Maximum Likelihood Estimators (MLE) and the Maximum Product of Spacings (MPS) for the Logistic-Exponential (LE) distribution.

5.1 Synthesis of Simulation Findings

The experimental data summarized in Tables (1) through (5) provide robust evidence of the statistical consistency inherent in both estimation paradigms. A monotonic decrease in Mean Squared Error (MSE) values is observed as the sample size expands from $n = 25$ to $n = 100$. Such behavior indicates that the estimators converge toward the true parameter values in probability as the informational density of the sample improves, which aligns with the asymptotic efficiency frameworks established by Ekström et al. (2019) and Ali et al. (2025).

In terms of methodological divergence, the results underscore a clear precision advantage for the MPS approach in identifying the shape parameter (κ), particularly under restricted sample conditions ($n = 25$). For instance, in the fifth scenario (Table 5) where $\kappa = 2.5$, MPS yielded a significantly lower MSE of 0.2088 compared to 0.2650 for MLE. This superior stability is attributable to the bounded nature of the spacings function, which provides a protective buffer against the numerical instabilities often encountered by the likelihood function in highly skewed distributions or at the boundaries of the parameter space (Cheng & Amin, 1983; Nassar et al., 2023).

Furthermore, the simulation revealed that the accuracy of the scale parameter (λ) is intrinsically linked to the underlying shape configuration. The rising MSE trend as the system moves toward high-hazard scenarios ($\kappa = 2.5$) highlights a heightened model sensitivity, necessitating the deployment of robust estimators like MPS to mitigate potential biases resulting from data concentration in the distribution's tails (Alslman & Helu, 2022; Jeon et al., 2022).

5.2 Analysis and Interpretation of Empirical Findings

1. Model Adequacy and Flexibility:

Results from Chi-square test and Anderson-Darling test (Table 4) show high flexibility of the Logistic-Exponential (LE) distribution. For both empirical estimation methods, the values of χ^2 are smaller than critical values and corresponding high P-values ($P \geq 0.05$) justify the use of the Logistic-Exponential model to model stochastic behavior of component failure times. Since LE model is flexible and easy to fit (Lan and Leemis 2008), it serves as an alternative to the well known classical survival models for analyzing IFR data.

2. Superiority of MPS in Empirical Estimation:

Both (MPS) and (MLE) methods provided good fits. As can be seen from the tables, the χ^2 values for the MLE method are only slightly larger (2.5251) than those (2.0075) obtained with the MPS method. For small samples (n = 23), in particular, the MPS method is handle the issue of spacing of observed values more efficiently.

3. Alignment with Simulation Outcomes:

Significantly, both the simulation study and the real data application produced analogous conclusions. The simulation results (from Scenario III & IV) showed that as $\kappa>1$, the efficiency of MPS method maintained its level of efficiency superiority. This finding is in line with the characteristics of real data that exhibits an Increasing Failure Rate (IFR) trend. The synchronization between simulated results and real data further reinforces the applicability and integrity of the developed estimation framework for the purposes of statistical inference.

6. Conclusion and Recommendations

6.1 Concluding Remarks

In this article, we implemented a comparative study between Maximum Likelihood Estimation (MLE) and Maximum Product of Spacings (MPS) methods for estimating parameters of the Logistic-Exponential (LE) distribution. Results of the simulation study and real data application were combined and discussed:

* **Methodological Superiority:** The study systematically demonstrates that MPS outperform MLE in terms of estimation accuracy. In particular, the study establishes that MPS consistently outperform MLE in terms of Mean Squared Error (MSE). The methodological superiority of MPS is shown to be particularly pronounced in small sample sizes (n=25) as MPS systematically manage probability gaps across ordered observations to consistently deliver precise parameters.

* **Model Versatility:** The Logistic-Exponential distribution proved to be a very versatile model for the reliability analysis of the given data set. Due to its shape characteristics, it is able to fit the given data very well, especially in terms of the hazard rate function where it can capture increasing failure rate (IFR) behavior, and outperform other competitors such as the Weibull or Gamma distributions for the given real data set.

* **Statistical Consistency:** Both estimation methods are asymptotically consistent, meaning that as sample size goes to infinity, the difference between the MLE and MPS goes to zero, and both estimators consistently recover the true parameters.

* **Empirical Validation:** The application to real-world failure data confirmed the simulation outcomes. The MPS method achieved a superior goodness-of-fit ($\chi^2 = 2.0075, P = 0.57085$), validating the practical utility of spacing-based estimation in engineering and survival analysis, consistent with the reliability assessment frameworks discussed in Samih and Alwan (2024).

6.2 Recommendations

Based on the synthesized results, the following recommendations are proposed for researchers and practitioners:

1. Adoption of MPS for Limited Data: In reliability engineering and biomedical applications where the sample size is small or even costly to obtain, we prefer to use MPS over classical MLE to achieve better estimation accuracy.

2. Implementation of LE Distribution: For data with monotonic or non-monotonic hazards, the Logistic-Exponential model is a best fit within the LE family due to its excellent performance in terms of goodness-of-fit and its flexibility to obtain a wide range of shapes.

3. Future Research Directions:

- The analysis could also be extended to Censored Data (Type-I and Type-II). Typically survival data are only available in incomplete form in real-world applications.

- Further exploration of Bayesian Estimation for the LE distribution using informative and non-informative priors could provide an interesting comparison to the frequentist approaches presented.

- The LE model and its extensions can be further applied to other complex systems in real-world applications such as environmental reliability analysis as discussed in Samih and Alwan.(٢٠٢٤)

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