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New Topp Leone- Shanker distribution: Properties and Estimation

توزيع توب ليون- شانكر الجديد: الخصائص والتقدير

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Abstract : *In this paper, the researchers introduce a new two-parameter distribution called the Topp-Leone-Shanker distribution. We then expand the distribution and rewrite both the function c.d.f and the function p.d.f for the proposed distribution. The study in some of its mathematical properties represented by [The Non-Central Moment and The Central Moment, C.S, C.K, C.V], in addition to estimating the distribution parameters using the maximum likelihood estimation method (MLE), the Maximum product of spacing Estimation Method (MPS) and compere them using simulation, using four sample sizes (20,35,75,125) and then finding the maximum product of spacing estimation method to be preferable to the maximum likelihood method of estimation due to having the lowest mean squared error (MSE).*

Keywords: *Topp-Leone Shanker, Estimation, Simulation.function c.d.f .function p.d.f*

الملخص:في هذه الورقة، سيقدم الباحثان توزيعاً جديداً يتكون من معلمتين ويسمى توزيع توب ليون- شانكر. ثم نقوم بتوسيع التوزيع وإعادة كتابة كل من دالة التوزيع التراكمي CDF ودالة الكثافة الاحتمالية PDF لتوزيع المقترح. وندرس قسما من الخصائص الإحصائية للتوزيع المقترح مثل العزم المركزي الرائي والعزم اللامركزي الرائي ومعاملي الالتواء والتفلطح ومعامل الاختلاف. بالإضافة الى تقدير معلمات التوزيع باستخدام طريقة الإمكان الأعظم (MLE) وطريقة الحد الأقصى الناتج لمقدرات التباعد (MPS). ومقارنتها باستخدام المحاكاة باستعمال أربعة حجوم عينات (20,35,75,125). وتم التوصل الى افضلية طريقة الحد الأقصى لمقدرات التباعد على طريقة الإمكان الأعظم في التقدير نتيجة لامتلاكها اقل متوسط مربعات خطأ.

الكلمات المفتاحية: توب ليون- شانكر، تقدير، المحاكاة.دالة التوزيع التراكمي ،دالة الكثافة الاحتمالية

مجلة علمية فصلية محكمة تعنى بالشؤون الاقتصادية والإدارية والمحاسبية والمالية والإحصائية للخليج العربي والجزيرة العربية تصدر عن مركز دراسات البصرة والخليج العربي جامعة البصرة

1. Introduction

Statistical distributions are widely used in the study of natural phenomena, and over the years many distributions have been identified and studied, however, they have a limited range of capabilities, so they cannot be used in all cases. This is due to the fact that the characteristics of the phenomenon change over time, and as a result, the statistical researchers face many issues and difficulties during the statistical analysis process, and at the forefront of these difficulties is determining the probability distribution represented by the high degree of twisting and flattening or the limited appearance of data values on the right side without the left or vice versa. In statistics, there are many probability distributions that have been recognized and studied and have been entered in many areas of life using real data, and we may face in some data issues to obtain realistic results due to the reason that the probability distribution does not have the ability to dub the data, so other methods such as (mixing or fitting) distributions are used. Many Studies and researches have dealt with the family of Topp Leone Shanker in building new probabilistic models, including: In 2016, (Al- Shomrani, et al.) [9] they have proposed a new family of distributions; the Topp–Leone family of distributions. They have given general expression for density and distribution function of the new family. Expression for moments and hazard rate has also been given. They have also given an example of the proposed family. And In 2017, (Brito, et al.) [5] family distribution (Topp Leone add Log-Logistic) proposed a one-parameter and then worked on studying its structural characteristics [The Non-Central Moment and The Central Moment and C.V and C.K and C.S], the researchers also estimated the Parameters of the distribution using the Maximum Likelihood Method as well as employ a simulation method to show the effect of sample size on the maximum likelihood estimates of the parameters and then compared the estimates of a study distribution with some other distribution based on real data better than other distribution included in a study. Also In 2015, (Shanker) [8] a new one parameter lifetime distribution named “Shanker distribution” for modeling lifetime data has been suggested. Various mathematical properties of the new distribution including its shape, moments, generating functions, hazard rate and mean residual life functions, stochastic ordering, order statistics, Renyi entropy measure, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have been presented. The conditions of over-dispersion, equi-dispersion, and under-dispersion of Shanker distribution have been discussed. The method of maximum likelihood estimation and the method of moments have been discussed for estimating its parameter. The usefulness, applicability and superiority of the proposed distribution over exponential and Lindley distributions have been illustrated with three real lifetime data- sets. And In 2021 (Ganaie et al.) [7] we propose a new extension of power Shanker distribution known as weighted power Shanker distribution. The different mathematical and statistical properties of the newly introduced distribution are derived and discussed in detail such as moments, order statistics, likelihood Ratio test, Income distribution curves, and entropy and reliability measures. The maximum likelihood estimators of the parameters of new distribution are executed and also the Fisher's information matrix is discussed. Application of the new distribution with three real life data sets are executed to show the supremacy of weighted power shanker distribution in analyzing real lifetime data. This Research distinct from previous studies in that it worked on proposing a probability distribution as a new member of the Topp Leone - Shanker presented by (Shomrani et al.) in (2016), in to studying it is statistics properties and estimating it is parameters by the (MLE) Method and the (MPS) Method. This paper differs from the literature in that it is concerned with constructing a new probability distribution using the Topp Leone- Shanker family, the two parameter (θ, α), deriving its structural properties and estimating its parameters in two ways (MLE, MPS).

1. Proposed Distribution Topp Leone Shanker (TL-Sh)

1.1. Topp Leone Family

The Topp-Leone distribution is a continuous statistical distribution that has many uses in analyzing survival, reliability, and failure functions. Introduced by (Topp and Leone,1955), the probability density function of this distribution is characterized by a J-shaped shape, as well as an alternative to the beta distribution. The probability density function and Topp-Leone distribution are defined by the equations (1), (2) below:

$f_{TL}(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}$; $0 \le x \le 1, \alpha > 0$ ... (1)

$F_{TL}(x) = x^{\alpha}(2-x)^{\alpha}$; $0 \le x \le 1, \alpha > 0$ ... (2)

The Topp Leone family has been suggested by (AL-Shomrani et al., 2016), [9] the family's probability and distribution density function are defined by the equations (3), (4):

$f_{TL}(x) = 2\alpha g(x)\bar{G}(x)[1-\bar{G}(x)^2]^{\alpha-1}$ ... (3)

α: Shape parameter

$g(x)$: The probability density function of the basic distribution

The cumulative function of the Topp Leone family is defined by the following equation:

$F_{TL}(x, \alpha) = [1-(\bar{G}(x))^2]^{\alpha}$; $x > 0$, $\alpha > 0$ … (4)

Where

$\bar{G}(x) = 1 - G(x)$

$G(x)$: the cumulative function of the basic distribution

2.2 Shanker distribution

Distribution the Shanker, is a continuous probability distribution proposed By (Shanker, 2015) to represent survival data, which is produced by mixing the exponential distribution with parameter (θ) and the gamma distribution with two parameters (2,θ),The cumulative function of the Shanker distribution is known as equation (5): [8]

$$G(x,\theta) = 1 - \left[\frac{(\theta^2+1)+\theta x}{\theta^2+1}\right] e^{-\theta x}; x > 0, \theta > 0 \qquad \dots (5)$$

As for the probability density function, it is defined in equation (6) below:

$$g(x,\theta) = \frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\ ; x > 0, \theta > 0 \qquad \dots (6)$$

The reliability and hazard functions are defined by equations (8) and (7):

$$R(x,\theta) = \left(\frac{(\theta^2+1)+\theta x}{\theta^2+1}\right) e^{-\theta x} \qquad \dots (7)$$

$$h(x,\theta) = \frac{g(x,\theta)}{R(x,\theta)}$$

$$h(x,\theta) = \frac{\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}}{\left(\frac{(\theta^2+1)+\theta x}{\theta^2+1}\right) e^{-\theta x}} \qquad \dots (8)$$

2.3 Topp Leone-Shanker Proposed Probability Distribution (TL-Sh)

A New Probability Model will be Proposed by (the Topp-Leone family) and the Shanker Distribution to make the resulting distribution a new member of this family, expressed by the symbol (TL-Sh), through the following steps:

Substituting equation (7) into equation (4) and simplifying, we obtain the cumulative distributional function of the Topp Leone-Shanker distribution as in equation (9):

$$G_{TL-Sh}(x;\alpha,\theta) = 1 - \left[\left(\frac{(\theta^2+1)+\theta x}{\theta^2+1}\right)^2\right]^{\alpha} \qquad \dots (9)$$

To obtain the probability density function of the Topp Leone-Shanker distribution, we substitute equations (7) and (6) into equation (3) and shown in equation (10):

$$g_{TL-Sh}(x;\alpha,\theta) = 2\alpha \left[\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\right]\left[\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right]$$

$$\left(\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right)^2 \ [1-]^{\alpha-1} x > 0, \theta > 0 \qquad \dots (10)$$

$g_{TL-Sh}(x;\alpha,\theta)$:The probability density function (p.d.f.) of the proposed model (TL-Sh).

$G_{TL-Sh}(x;\alpha,\theta)$: The cumulative distribution function (c.d.f.) of the proposed model (TL-Sh).

α: Shape parameter

θ: *Scale Parameter*

1. Structural Properties

3.1 The Non-Central Moment

The Non-Central moment of the (TL-Sh) distribution of degree r[th] is defined by equation (11) and (12) below:

$$\acute{\mu} = E(X^r) = \int_0^\infty x^r \, g_{TLSh}(x,\alpha,\theta)dx \qquad \dots (11)$$

Using the binomial theorem and simplifying, we get

$$\acute{\mu}_r = \frac{-2\alpha\theta^2}{(\theta^2+1)^2} \sum_{k=0}^{\alpha-1} \binom{\alpha-1}{k} \int_0^\infty \frac{y^r}{2^r\theta^r} \left(\frac{(2\theta^2+y)(2(\theta^2+1)+y)}{4\theta} \right) \left(\frac{(2(\theta^2+1)+y)}{2(\theta^2+1)} e^{-\frac{y}{2}} \right)^{2(\alpha-(k+1))} e^{-y} \frac{1}{2\theta} \qquad \dots (12)$$

3.2 The Central Moment

Below is a definition of the mathematical formula for the μ_r equation in (13) of the (TL-Sh) distribution:

$$\mu_r = E(X-\mu)^r = \int_0^\infty (x-\mu)^r \, g_{TLSh}(x,\alpha,\theta)dx \qquad \dots (13)$$

Using the binomial theorem and simplifying, we get

$$\mu_r = \frac{2\alpha\theta^2}{(\theta^2+1)^2} \sum_{i=0}^{r} \sum_{k=0}^{\alpha-1} \binom{r}{i} \binom{\alpha-1}{k} (-\mu)^{r-i} \int_0^\infty x^i \left(\frac{(2\theta^2+y)(2(\theta^2+1)+y)}{4\theta} \right) \left(\frac{(2(\theta^2+1)+y)}{2(\theta^2+1)} e^{-\frac{y}{2}} \right)^{2(\alpha-(k+1))} e^{-2\theta x} \, dx \qquad \dots (14)$$

3.3 Skewness Coefficient

We are working to derive the mathematical formula for the S.K as follows in equation (15); [2]

$$C.S = \frac{\mu_3}{\sqrt{(\mu_2)^3}} \qquad \dots (15)$$

By Substituting the second central moment μ_2 and the third central moment μ_3 in equation (15), we obtain the following coefficient Skewness.

3.4 Kurtosis Coefficient

With regards to coefficient kurtosis C.K it is mathematical form is known by the following equation (16), [4]

$$C.K = \frac{\mu_4}{(\mu_2)^2} \qquad \dots (16)$$

By Substituting, the second central moment (μ_2) and the third central moment (μ_4) into the above equation; we obtain the following coefficient kurtosis.

3.5 Variation Coefficient

The coefficient of variation is according by equation (17): [3]

$$C.v = \frac{\sqrt{\mu_2}}{\acute{\mu}_1} \qquad \dots (17)$$

Substituting $\acute{\mu}_1$ and μ_2 into formula (17) we get

4.Estimation

4.1. Maximum likelihood Estimation (MLE)

We (X) is assumed to be a random variable that follows the (TL-Sh) distribution then the maximum possibility function represents, the joint function of the independent random variable $(x_1, x_2, x_3 \dots, x_n)$ as follows: [6],[10]

$$L(x_1, \dots, x_n, \alpha, \theta) = \prod_{i=1}^{n} f_{TLSh}\,(x_i, \alpha, \theta) \qquad \dots (18)$$

$$L(x_1, \dots, x_n, \alpha, \theta) = \prod_{i=1}^{n} 2\alpha \left[\frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x}\right]\left[\frac{\left((\theta^2+1)+\theta x\right)}{\theta^2+1}e^{-\theta x}\right] \left[1-\left(\frac{\left((\theta^2+1)+\theta x\right)}{(\theta^2+1)}e^{-\theta x}\right)^2\right]^{\alpha-1}$$

We get the natural log of both sides of the equation,

$$\begin{aligned}\ln L &= n\ln 2\alpha + n\ln\theta^2 - n\ln(\theta^2+1) + \sum_{i=1}^{n}\ln(\theta + x_i) - \theta\sum_{i=1}^{n} x_i \\ &\quad - n\ln(\theta^2+1) + \sum_{i=1}^{n}\ln(\theta^2+1+\theta x_i) - \theta\sum_{i=1}^{n} x_i \\ &\quad + (\alpha \\ &\quad - 1)\sum_{i=1}^{n}\ln\left(1-\left(\frac{\left((\theta^2+1)+\theta x\right)}{(\theta^2+1)}e^{-\theta x}\right)^2\right) \qquad \dots (19)\end{aligned}$$

We take the partial derivative of equation (19) and set it equal to zero for the parameters (α, θ):

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1-\left(\frac{\left((\theta^2+1)+\theta x\right)}{(\theta^2+1)}e^{-\theta x}\right)^2\right) = 0 \qquad \dots (20)$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &= \frac{2n}{\theta} - \frac{4n\theta}{\theta^2+1} + \sum_{i=1}^{n}\frac{1}{\theta + x_i} - 2\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{2\theta + x_i}{\theta^2+1+\theta x_i} \\ &\quad - (\alpha-1)\sum_{i=1}^{n}\frac{2(A*B)\left(A*\frac{\partial B}{\partial\theta} + B\frac{\partial A}{\partial\theta}\right)}{[1-(A*B)^2]} \\ &= 0 \qquad \dots (21)\end{aligned}$$

Where

$$A = \frac{(\theta^2 + 1) + \theta x}{\theta^2 + 1}$$

$$\frac{\partial A}{\partial x} = \frac{(\theta^2 + 1 + \theta x_i)(2\theta) - (\theta^2 + 1)(2\theta + x_i)}{(\theta^2 + 1)^2} \qquad \dots (22)$$

$$B = e^{-\theta x_i} \quad ; \quad \frac{\partial B}{\partial \theta} = x_i e^{-\theta x_i} \qquad \dots (23)$$

We note that equations (20) and (21) are non-linear equations, and therefore cannot be solved by the classical mathematical methods, so we will resort to finding the value of the parameters by numerical methods, including the (Newton-Raphson) method to obtain MLE for the unknown parameters $\left(\hat{\alpha}_{mle}, \hat{\theta}_{mle} \right)$.

4.2. Maximum product of spacing estimation method (MPS)

This method is used to estimate the parameters when the data is continuously distributed, this method is an alternative to the maximum likelihood method (MLE) for estimating unknown continuous univariate parameters, what distinguishes the (MPS) method is that it remains consistent and more efficient in some cases than the (MLE) method presented by (Cheng, Amin ,1983) and for the finding of the parameter estimators for the proposed model (TL-Sh): [1]

$$K(\alpha, \theta) = \left(\prod_{i=1}^{m+1} D_i(\alpha, \theta) \right)^{\frac{1}{m+1}} \qquad \dots (24)$$

Where

$$D_i(\alpha, \theta) = G_{TLSh}(x_i, \alpha, \theta) - G_{TLSh}(x_{i-1}, \alpha, \theta)$$

$$\sum_{i=1}^{m+1} D_i(\alpha, \theta) = 1 \qquad , i = 1,2, \dots \dots, m+1$$

$$D_i(\alpha, \theta) = \begin{cases} G(x_i) & if \ i = 1 \\ G(x_i) - G(x_{i-1}) & if \ i = 1,2, \dots, m \\ 1 - G(x_i) & if \ i = m+1 \end{cases}$$

Taking the natural log of equation (24) gives

$$\ln K(\alpha, \theta) = \frac{1}{m+1} \sum_{i=1}^{m+1} D_i(\alpha, \theta)$$

$$\ln K(\alpha, \theta) = \frac{1}{m+1} \left[\ln D_1(\alpha, \theta) + \sum_{i=2}^{m} \ln D_i(\alpha, \theta) + \ln D_{m+1}(\alpha, \theta) \right]$$

$$\ln K(\alpha,\theta) = \frac{1}{m+1}\left[\ln G(x_i) + \sum_{i=2}^{m} \ln\big(G(x_i) - G(x_{i-1})\big) + \ln\big(1 - G(x_i)\big)\right] \quad \dots (25)$$

Substituting equation (9) into (25) gives

$$\ln\ln K(\alpha,\theta) = \frac{1}{m+1}\left[\ln\left(\left(\frac{(\theta^2+1)+\theta x_1}{(\theta^2+1)}e^{-\theta x_1}\right)^2\right)^{\alpha} + \sum_{i=2}^{m}\ln\left(\left(\left(\frac{(\theta^2+1)+\theta x_i}{(\theta^2+1)}e^{-\theta x_i}\right)^2\right)^{\alpha} - \left(\left(\frac{(\theta^2+1)+\theta x_{i-1}}{(\theta^2+1)}e^{-\theta x_i-1}\right)^2\right)^{\alpha}\right) - \ln\left(1 - \left(\left(\frac{(\theta^2+1)+\theta x_n}{(\theta^2+1)}e^{-\theta n}\right)^2\right)^{\alpha}\right)\right] \quad \dots (26)$$

$$= \frac{1}{m+1}\left[((A_1B_1)^2)^{\alpha} + \sum_{i=2}^{m}\ln(((A_2B_2)^2)^{\alpha} - ((A_3B_3)^2)^{\alpha}) + \ln(1 - ((A_4B_4)^2)^{\alpha})\right] \quad ..(27)$$

Where

$$A_1 = \frac{((\theta^2+1)+\theta x_1)}{(\theta^2+1)} \quad ; B_1 = e^{-\theta x_1}$$

$$A_2 = \frac{(\theta^2+1)+\theta x_i}{(\theta^2+1)} \quad ; B_2 = e^{-\theta x_i}$$

$$A_3 = \frac{(\theta^2+1)+\theta x_{i-1}}{(\theta^2+1)} \quad ; B_3 = e^{-\theta x_{i-1}}$$

$$A_4 = \frac{(\theta^2+1)+\theta x_n}{(\theta^2+1)} \quad ; B_4 = e^{-\theta x_n}$$

To find the divergence estimators for the TL-Sh distribution, we derive equation (27) for the parameters (α, θ), and then set it equal to zero:

First, for parameter (α):

$$\frac{\partial K(\alpha,\theta)}{\partial \alpha} = \frac{1}{m+1}\left[\frac{\partial}{\partial \alpha}\ln[((A_1B_1)^2)^\alpha] + \frac{\partial}{\partial \alpha}\sum_{i=2}^{m}\ln(((A_2B_2)^2)^\alpha - ((A_3B_3)^2)^\alpha) + \frac{\partial}{\partial \alpha}\ln(1-((A_4B_4)^2)^\alpha)\right] \quad \ldots (28)$$

$$\frac{\partial}{\partial \alpha}((A_1B_1)^2)^\alpha = \frac{\partial}{\partial \alpha}(2\alpha\ln A_1B_1) = 2\ln A_1B_1$$

$$\frac{\partial}{\partial \alpha}\ln(((A_2B_2)^2)^\alpha - ((A_3B_3)^2)^\alpha) = \frac{1}{((A_2B_2)^2)^\alpha - ((A_3B_3)^2)^\alpha} * \left(e^{-\alpha(\ln A_2B_2)^2} * \ln(A_2B_2)^2 * e^{-\alpha(\ln A_3B_3)^2} * \ln(A_3B_3)^2\right)$$

$$\frac{\partial}{\partial \alpha}[\ln(1-((A_4B_4)^2)^\alpha)] = -\frac{1}{[\ln(1-((A_4B_4)^2)^\alpha)]} * e^{-\alpha(\ln A_4B_4)^2} * \ln(A_4B_4)^2$$

Substituting the above derivatives into equation (28) gives

$$\frac{\partial K(\alpha,\theta)}{\partial \alpha} = \frac{1}{m+1}\left[(2\ln A_1B_1) + \left(\frac{1}{((A_2B_2)^2)^\alpha - ((A_3B_3)^2)^\alpha} * \left(e^{-\alpha(\ln A_2B_2)^2} * \ln(A_2B_2)^2 * e^{-\alpha(\ln A_3B_3)^2} * \ln(A_3B_3)^2\right)\right) - \left(\frac{1}{[\ln(1-((A_4B_4)^2)^\alpha)]} * e^{-\alpha(\ln A_4B_4)^2} * \ln(A_4B_4)^2\right)\right] \quad \ldots (29)$$

Second, for parameter (θ):

$$\frac{\partial K(\alpha,\theta)}{\partial \theta}=\frac{1}{m+1}\left[\frac{\partial}{\partial\theta}\ln[((A_1B_1)^2)^\alpha]\right.$$
$$+\frac{\partial}{\partial\theta}\sum_{i=2}^{m}\ln(((A_2B_2)^2)^\alpha-((A_3B_3)^2)^\alpha)$$
$$\left.+\frac{\partial}{\partial\theta}\ln(1-((A_4B_4)^2)^\alpha)\right] \quad \ldots(30)$$

$$\frac{\partial}{\partial\theta}\ln[((A_1B_1)^2)^\alpha]=\frac{2\alpha}{((A_1B_1)^2)^\alpha}((A_1B_1)^2)^{\alpha-1}A_1B_1\left(A_1\frac{\partial B_1}{\partial\theta}+B_1\frac{\partial A_1}{\partial\theta}\right)$$

$$\frac{\partial}{\partial\theta}\ln(((A_2B_2)^2)^\alpha-((A_3B_3)^2)^\alpha)$$
$$=\frac{1}{((A_2B_2)^2)^\alpha-((A_3B_3)^2)^\alpha}\left[2\alpha(A_2B_2)^{\alpha-1}*(A_2B_2)\right.$$
$$*\left(\left(A_2\frac{\partial B_2}{\partial\theta}+B_2\frac{\partial A_2}{\partial\theta}\right)\right)-2\alpha(A_3B_3)^{\alpha-1}*(A_3B_3)$$
$$\left.*\left(\left(A_3\frac{\partial B_3}{\partial\theta}+B_3\frac{\partial A_3}{\partial\theta}\right)\right)\right]$$

$$\frac{\partial}{\partial\theta}[\ln(1-((A_4B_4)^2)^\alpha)]$$
$$=-\frac{1}{[\ln(1-((A_4B_4)^2)^\alpha)]}$$
$$*\left[2\alpha(A_4B_4)^{\alpha-1}*(A_4B_4)*\left(\left(A_4\frac{\partial B_4}{\partial\theta}+B_4\frac{\partial A_4}{\partial\theta}\right)\right)\right]$$

Substituting the above derivatives into equation (*30*) gives

$$\frac{\partial K(\alpha,\theta)}{\partial\theta}=\frac{1}{m+1}\Bigg[\Bigg(\frac{2\alpha}{((A_1B_1)^2)^{\alpha}}((A_1B_1)^2)^{\alpha-1}A_1B_1\left(A_1\frac{\partial B_1}{\partial\theta}+B_1\frac{\partial A_1}{\partial\theta}\right)\Bigg)$$
$$+\left(\frac{1}{((A_2B_2)^2)^{\alpha}-((A_3B_3)^2)^{\alpha}}\Bigg[2\alpha(A_2B_2)^{\alpha-1}*(A_2B_2)\right.$$
$$*\left(\left(A_2\frac{\partial B_2}{\partial\theta}+B_2\frac{\partial A_2}{\partial\theta}\right)\right)-2\alpha(A_3B_3)^{\alpha-1}*(A_3B_3)$$
$$\left.*\left(\left(A_3\frac{\partial B_3}{\partial\theta}+B_3\frac{\partial A_3}{\partial\theta}\right)\right)\Bigg]\right)$$
$$+\left(-\frac{1}{[\ln(1-((A_4B_4)^2)^{\alpha})]}\right.$$
$$*\Bigg[2\alpha(A_4B_4)^{\alpha-1}*(A_4B_4)$$
$$\left.*\left(\left(A_4\frac{\partial B_4}{\partial\theta}+B_4\frac{\partial A_4}{\partial\theta}\right)\right)\Bigg]\right)\Bigg]\quad\ldots(31)$$

To obtain the parameter estimator by the maximum divergence product method $\hat{\alpha}_{MPS},\hat{\theta}_{MSP}$ for the proposed model (TL-Sh), we substitute the derivatives, (22), (23) into equations, (29), (31) and solve them by numerical methods, solved by Newton-Raphson method because they are nonlinear.

2. Simulation

The aim of obtaining the best estimation method is specified in the paper which are (MLE) and method (MPS) using the ranks method, the simulation was carried out where a rank is given to the methods based on the value of the mean square error (MSE) for each sample size and each model (PARTIAL Rank). The best method for all models is determined (OVERALL RANK), which is the lowest is the best method.

Based on four Sample, Size (20, 35, 75, and 125) and four models for Values of distribution parameters (TL-Sh) the simulation was carried out. Moreover, the simulations results in (1), (2), (3), (4) and (5) Below:

Table [1]; Estimation of parameters and mean squares of error at default values

$$(\alpha = 2.5\,,\theta = 0.5)$$

N	Estimate	Parm	Method			Best
20	Parm	MLE		PC	MPS	
		θ	0.51456	0.47529	0.44758	
		α	3.2769	2.5058	2.27176	
	MSE	θ	0.00860099[1]	0.00873649[2]	0.0104585[3]	MPS

		α	2.16536^{3}	1.02596^{2}	0.735295^{1}	
$\sum Rank$			4^{2}	4^{2}	4^{2}	
Partial Rank			2	2	2	
35	Parm	θ	0.53692	0.47436	0.490396	
		α	2.8575	2.12	2.25994	
	MSE	θ	0.00518466^{3}	0.00475056^{2}	0.00364716^{1}	
		α	0.694926^{3}	0.414291^{2}	0.379788^{1}	MPS
$\sum Rank$			6^{3}	4^{2}	2^{1}	
Partial Rank			3	2	1	
75	Parm	θ	0.52108	0.48266	0.491422	
		α	2.8007	2.114	2.39703	
	MSE	θ	0.00229245^{2}	0.00472145^{3}	0.00150535^{1}	
		α	0.405901^{3}	0.39565^{2}	0.159302^{1}	MPS
$\sum Rank$			$5^{2.5}$	$5^{2.5}$	2^{1}	
Partial Rank			2.5	2.5	1	
125	Parm	θ	0.48165	0.48945	0.46489	
		α	2.3891	2.1547	2.19244	
	MSE	θ	0.00211491^{2}	0.0043254^{3}	0.00112472^{1}	MPS
		α	0.238829^{2}	0.36482^{3}	0.11587902^{1}	
$\sum Rank$			4^{2}	6^{3}	2^{1}	
Partial Rank			2	3	1	
$\sum\sum Rank$			9.5	9.5	5	MPS
Overall Rank			2.5	2.5	1	
Best						

Table [2]; Estimation of parameters and mean squares of error at default values

Values ($\alpha = 2.5$, $\theta = 1$)

N	Estimate	Parm	Method			Best
			MLE	PC	MPS	
20	Parm	θ	1.024	0.87998	0.89551	
		α	2.6953	1.8735	1.92639	
	MSE	θ	0.03058^{1}	0.06294^{3}	0.035049^{2}	
		α	0.766138^{2}	0.859185^{3}	0.640358^{1}	MPS
$\sum Rank$			$3^{1.5}$	6^{3}	$3^{1.5}$	

Partial Rank			1.5	3	1.5	
35	Parm	θ	1.0527	0.9452	0.968825	MPS
		α	2.55545	2.4708	2.48435	
	MSE	θ	0.02754^{2}	0.0313^{3}	0.0239483^{1}	
		α	0.702545^{3}	0.650309^{2}	0.368637^{1}	
$\sum Rank$			$5^{2.5}$	$5^{2.5}$	2^{1}	
Partial Rank			2.5	2.5	1	
75	Parm	θ	1.0085	0.93844	0.951055	MPS
		α	2.655	2.2604	2.2894	
	MSE	θ	0.00897^{2}	0.02247^{3}	0.008929^{1}	
		α	0.367568^{3}	0.349684^{2}	0.252919^{1}	
$\sum Rank$			$5^{2.5}$	$5^{2.5}$	2^{1}	
Partial Rank			2.5	2.5	1	
125	Parm	θ	1.0073	0.93334	0.976863	MLS
		α	2.4935	2.1494	2.30438	
	MSE	θ	0.00575^{1}	0.0171^{3}	0.00575903^{2}	
		α	0.126419^{1}	0.32451^{3}	0.142908^{2}	
$\sum Rank$			2^{1}	6^{3}	4^{2}	
Partial Rank			1	3	2	
$\sum\sum Rank$			7.5	11	5.5	MPS
Overall Rank			2	3	1	
Best						

Table [3]; Estimation of parameters and mean squares of error at default values
Values $(\alpha = 2.5 , \theta = 1)$

N	Estimate	Parm	Method			Best
			MLE	PC	MPS	
20	Parm	θ	0.51269	0.46667	0.441661	PC
		α	1.9209	1.4847	1.39627	
	MSE	θ	0.00572321^{1}	0.00629072^{2}	0.00915458^{3}	
		α	0.660775^{3}	0.189077^{1}	0.248961^{2}	
$\sum Rank$			4^{2}	3^{1}	5^{3}	
Partial Rank			2	1	3	

35	Parm	θ	0.54971	0.50673	0.498548	MPS
		α	1.786	1.51247	1.44837	
	MSE	θ	0.0056222^{2}	0.00584332^{3}	0.00402025^{1}	
		α	0.258832^{3}	0.187487^{2}	0.11847^{1}	
$\sum Rank$			$5^{2.5}$	$5^{2.5}$	2^{1}	
Partial Rank			2.5	2.5	1	
75	Parm	θ	0.49813	0.50673	0.470272	MLE
		α	1.4772	1.2863	1.31463	
	MSE	θ	0.00201478^{1}	0.00444121^{3}	0.00278632^{2}	
		α	0.0390344^{1}	0.169477^{3}	0.0642545^{2}	
$\sum Rank$			2^{1}	6^{3}	4^{2}	
Partial Rank			1	3	2	
125	Parm	θ	0.49995	0.47776	0.469375	MLE
		α	1.4917	1.4384	1.38894	
	MSE	θ	0.00192215^{1}	0.00285478^{3}	0.0022214^{2}	
		α	0.02214887^{1}	0.139116^{3}	0.06213335^{2}	
$\sum Rank$			2^{1}	6^{3}	4^{2}	
Partial Rank			1	3	2	
$\sum\sum Rank$			6.5	9.5	8	MLE
Overall Rank			1	3	2	
Best						

Table [4]; Estimation of parameters and mean squares of error at default value Values $(\alpha = 1.5, \theta = 1)$

N	Estimate	Parm	Method			Best
			MLE	PC	MPS	
20	Parm	θ	1.0268	0.8711	0.890072	
		α	1.5822	1.1412	1.18709	

	MSE	θ	0.03575^{1}	0.07643^{3}	0.0398464^{2}	PC
		α	0.201996^{2}	0.252772^{3}	0.19274^{1}	
$\sum Rank$			$3^{1.5}$	6^{3}	$3^{1.5}$	
Partial Rank			1.5	3	1.5	
35	Parm	θ	1.0589	0.9397	0.971012	MPS
		α	1.8145	1.4847	1.49987	
	MSE	θ	0.03442^{2}	0.0384^{3}	0.0288168^{1}	
		α	0.195682^{2}	0.20178^{3}	0.0959661^{1}	
$\sum Rank$			4^{2}	6^{3}	2^{1}	
Partial Rank			2	3	1	
75	Parm	θ	1.0118	0.93321	0.946855	MPS
		α	1.5838	1.3698	1.38808	
	MSE	θ	0.01027^{2}	0.02721^{3}	0.0101825^{1}	
		α	0.106212^{2}	0.109248^{3}	0.0774394^{1}	
$\sum Rank$			4^{2}	6^{3}	2^{1}	
Partial Rank			2	3	1	
125	Parm	θ	1.0093	0.92798	0.976357	MPS
		α	1.491	1.307	1.3915	
	MSE	θ	0.0064^{1}	0.02124^{3}	0.00646443^{2}	
		α	0.0331859^{1}	0.0911903^{3}	0.0395871^{2}	
$\sum Rank$			2^{1}	6^{3}	4^{2}	
Partial Rank			1	3	2	
$\sum\sum Rank$			6.5	12	5.5	MPS
Overall Rank			2	3	1	
Best						

Table [5]; Estimation of parameters and mean of error at default values Values $(\alpha = 1, \theta = 0.5)$

N	Estimate	Param	Method			Best
			MLE	PC	MPS	
20	Parm	θ	0.5165	0.4232	0.437928	
		α	1.0601	0.76356	0.799774	
	MSE	θ	0.0115693^{1}	0.0266468^{3}	0.0128683^{2}	

		α	0.09029^{2}	0.11235^{3}	0.0825416^{1}	MLE -MPS
$\sum Rank$			$3^{1.5}$	6^{3}	$3^{1.5}$	
Partial Rank			1.5	3	1.5	
35	Parm	θ	0.53404	0.46409	0.484059	MPS
		α	1.2094	0.99047	1.00527	
	MSE	θ	0.0113578^{2}	0.0134399^{3}	0.00955678^{1}	
		α	0.084445^{2}	0.08906^{3}	0.0415108^{1}	
$\sum Rank$			4^{2}	6^{3}	2^{1}	
Partial Rank			2	3	1	
75	Parm	θ	0.50567	0.46059	0.476286	MLE -MPS
		α	1.0514	0.91354	0.945034	
	MSE	θ	0.00314102^{1}	0.00940442^{3}	0.00339906^{2}	
		α	0.04249^{2}	0.04834^{3}	0.0342721^{1}	
$\sum Rank$			$3^{1.5}$	6^{3}	$3^{1.5}$	
Partial Rank			1.5	3	1.5	
125	Parm	θ	0.50544	0.45656	0.486441	MLE -MPS
		α	0.99485	0.87086	0.928936	
	MSE	θ	0.00209652^{2}	0.00758325^{3}	0.00204913^{1}	
		α	0.01439^{1}	0.04133^{3}	0.0168779^{2}	
$\sum Rank$			$3^{1.5}$	6^{3}	$3^{1.5}$	
Partial Rank			1.5	3	1.5	MPS
$\sum\sum Rank$			6.5	12	5.5	
Overall Rank			2	3	1	
Best						

From the simulation; results in the tables 1,2,3,4,5 . Show that the Maximum product of spacing estimation (MPS) better than The Maximum Likelihood Estimation Method because, it has the least 7 option for the mean square error of the samples size and all default values of the parameters.

6. Conclusions

The Distribution of the study, Topp Leone Shanker family (TL-Sh) which is the proposed distribution, is highly flexible in fitting random variable data with complex behavior because it has good properties due to the inclusion of the shape parameter of the Topp Leone Shanker family as well as the derivation of the structural properties of the Topp Leone- Shanker [The Non-Central Moment and The Central Moment, Skewness Coefficient, Kurtosis Coefficient, Variation Coefficient]. The advantage of the (MPS) in estimating parameters for all samples sizes and all hypothetical models over and samples sizes the (MLE).

7. Recommendation

The researchers recommend the use of Bayesian methods in the estimation of the parameters of the TL-Sh distribution, as well as in the estimation of disease survival times, especially for patients with cancer.

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