ZANCO Journal of Pure and Applied Sciences

TYPE Original article
DOI http://dx.doi.org/10.21271/ZJPAS.38.3.4

OPEN ACCESS

*Corresponding author

Abdulqader Othman Hamadameen
abdulqader.othman@koyauniversity.org

RECEIVED :18/06/2025
ACCEPTED :16/11/ 2025
PUBLISHED :30/06/ 2026

KEYWORDS:

Fully fuzzy linear programming problem, fully fuzzy network, fuzzy maximal flow problem, fuzzy solution algorithm via Ford-Fulkerson method and linear ranking function, ranking function, and triangular fuzzy numbers.

Optimizing Fuzzy Maximum Flow Problems Using the Ford-Fulkerson Method and a Linear Ranking Function

Abdulqader Othman Hamadameen

Department of Mathematics, Faculty of Sciences and Health, Koya University, Danielle Mitterrand Boulevard, Koya KOY45, Kurdistan Region – F.R. Iraq

ABSTRACT

This study formulates the Fuzzy Maximum Flow Problem (FMFP) within Fully Fuzzy Networks (FFNs) as a Fully Fuzzy Linear Programming (FFLP) model, where all components—including objectives, constraints, and decision variables—are represented using Triangular Fuzzy Numbers (TrFNs). To address the inherent uncertainty and computational challenges, we propose a novel solution methodology: the Fuzzy Solution Algorithm via Ford-Fulkerson Method and Linear Ranking (FSA-FFMLR). This hybrid approach integrates a revised simplex method and Gaussian elimination within a linear ranking framework, transforming the original FFLP into a Semi-Fully Fuzzy Linear Programming (SFFLP) problem and enabling the efficient derivation of fuzzy optimal solutions. Computational experiments validate the robustness and efficiency of FSA-FFMLR, achieving an optimal fuzzy flow cost and outperforming benchmark methods in solution quality and convergence speed. The proposed methodology offers significant potential for real-world industrial applications, particularly in logistics, supply chain optimization, and infrastructure planning under uncertainty. Future research will focus on scaling the algorithm for large-scale decision-making environments to further enhance organizational resilience and adaptability.

1.Introduction

Network flow problems frequently arise in real-world situations and are crucial to operations research, computer science, and engineering (Li et al., 2013). In our daily lives, networks are everywhere we look. (Simpson, 2013) Through telephone networks, we may connect with one another virtually immediately both within and beyond national and international borders (Gandal et al., 2003). In general, network flow problems have been extensively researched in graph theory due to their wide-ranging applications across various fields, including telecommunications (Anttalainen, 2003), transportation (Bast et al., 2016), computing (Fitzek et al., 2020), manufacturing (Cheng et al., 2015, Cheng et al., 2011), and more.

The maximum flow problem is a key challenge in network flow analysis, aiming to transfer the largest possible flow between a source node (s) and a destination node (t) while respecting the capacity constraints of each arc. In conventional (crisp) scenarios, various efficient algorithms have been developed to solve this problem (Ahuja et al., 1993, Bazaraa et al., 2011, Hernandes et al., 2007). However, certain cases involve uncertainties in parameters such as costs, capacities, and demands. These cases are referred to as fuzzy maximum flow problems, where the network structure remains precise, but the parameters are represented with uncertainty.

The body of literature on the fuzzy maximum flow problem is relatively limited (Chanas and Kołodziejczyk, 1982, Chanas and Kołodziejczyk, 1984, Chanas and Kolodziejczyk, 1986, Kim and Roush, 1982). One of the earliest contributions to this field was the paper by Kim and Roush (Kim and Roush, 1982), in which they introduced fuzzy flow theory. Their work outlined the conditions necessary to achieve an optimal flow by leveraging definitions from fuzzy matrices. However, the most significant advancements in this area were made by Chanas and Kolodziejczyk (Hernandes et al., 2007), who pioneered key research on the topic. They tackled the problem using the minimum cuts technique, offering a different approach to addressing uncertainties in network flows. These studies were focused on the challenge of determining optimal flow within networks while ensuring arc capacity constraints are not excessively violated. In large-scale networks, identifying precise flow distribution becomes more difficult. Existing approaches present the maximum flow and minimum cut arcs but do not specify the flow quantity in each arc. To address this issue, the paper introduces an iterative algorithm designed to solve the maximum flow problem in graphs with crisp structures and fuzzy capacities (Bozhenyuk et al., 2017).

Uncertainties are handled using fuzzy set theory (Zadeh, 1965, Zadeh, 1996, Hamadameen, 2018, Hamadameen and Hassan, 2018b, Hamadameen and Hassan, 2018a, Hamadameen and Zainuddin, 2015). Chanas et al. investigated the maximum flow problem in networks with uncertain structures (Chanas et al., 1995, Chanas and Kołodziejczyk, 1982). Their study built upon previous work by Chanas and Kolodzijczyk (Chanas and Kołodziejczyk, 1982, Chanas and Kołodziejczyk, 1984, Chanas and Kolodziejczyk, 1986) and Delgado et al. (Delgado et al., 1985), which introduced the concepts of fuzzy maximum flow and fuzzy transportation (Hernandes et al., 2007).

Due to the effectiveness of fuzzy maximum flow in handling most networks and its capability to provide practical solutions, researchers have been inspired to develop methods and algorithms to address many of life's challenges(Kumar et al., 2009, Akram et al., 2022, Hernandes et al., 2007, Kumar and Kaur, 2012, Bozhenyuk et al., 2012, Kaur, 2010, Ekanayake et al., 2022, Goldberg and Tarjan, 1988, Kumar and Kaur, 2011, Majumder et al., 2018). This breakthrough has expanded possibilities, allowing complex problems to be utilized in ways that benefit humanity across various fields. There are some noteworthy works where researchers have successfully employed the Ford-Fulkerson method to address network problems in some vital areas (Ibrahim and Zebari, 2025, Davies et al., 2023, Kadhim, 2025).

This study presents a way to model FMFP within FFN using FFLP, where all components of objectives, constraints, and variables are expressed as the T_rFNs. It introduces a novel solution approach, the FSA-FFMLR, which combines a modified simplex method and Gaussian elimination within a ranking framework to transform the original FFLP into a SFFLP problem, making it possible to derive a fuzzy solution. Computational tests confirm the practicality and efficiency of this method. The remainder of this paper is structured as follows. Section II introduces fundamental fuzzy concepts, explores the algebraic properties of Triangular Fuzzy Numbers (T_rFNs), and examines the comparisons between different Fuzzy Numbers (FNs). Section III explores challenges related to Fully Fuzzy Networks (FFN) and Fuzzy Maximum Flow (FMF) problems. Section IV focuses on the formulation of the FMF problem as a Fully Fuzzy Linear Programming (FFLP) problem. Section V presents the Ford-Fulkerson fuzzy method for addressing FMF problems within the framework of the FFLP. Additionally, it outlines a procedure for solving FFLP problems using the Fuzzy Solution Algorithm via Ford-Fulkerson Method and Linear Ranking (FSA-FFMLR). In Section VI, the three step procedure of the FSA-FFMLR is applied to a numerical example to illustrate its practicality in solving FFLP problems. Section VII evaluates the results obtained and provides a structured justification for the FSA-FFMLR. Section VIII highlights the advantages of the proposed approach in handling uncertainty within FMF problems. Conclusions are discussed in Section IX.

2. Preliminaries of Fuzzy Concepts

In this section, we explore various types of fuzzy numbers, the algebraic properties of T_rFNs, and the linear ranking function along with its applications. These concepts are essential and play a crucial role in the subsequent sections of the research.

2.1. Kinds of Fuzzy Numbers

Definition1: Fuzzy Numbers. Fuzzy numbers have a continuous range of membership grades (Hamadameen and Hassan, 2018a) and are distinguished using a formula that gives each integer a grade within a range of the closed unit interval (Mahdavi-Amiri and Nasseri, 2006). Trapezoidal fuzzy number (T_pFN)s and triangular fuzzy number (T_rFN)s are the two types of fuzzy numbers that are most frequently seen (Ganesan and Veeramani, 2006, Mahdavi-Amiri and Nasseri, 2007, Dubois and Prade, 1978).

Definition2: The Trapezoidal Fuzzy Number (T_pFN). A T_pFN is $\tilde{A}$ = (a^L, a^U, α, β), where [a^L, a^U] is the modal set of $\tilde{A}$, and [$a^L - \alpha$, $a^U + \beta$] its support part (Hamadameen and Hassan, 2018a, Mahdavi-Amiri and Nasseri, 2006, Ganesan and Veeramani, 2006, Mahdavi-Amiri and Nasseri, 2007) as pictured in Figure1.

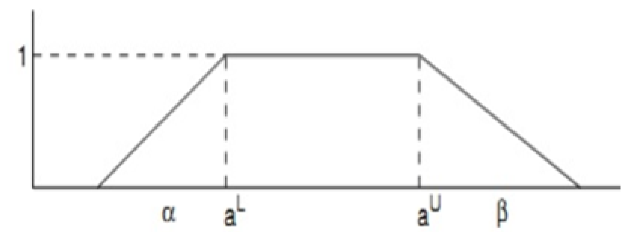

Figure1: T_pFN

Definition 3: The Triangular Fuzzy Number (T_rFN). If $a = a^L = a^U$ then the $TpFN$ is reduced to T_rFN, also indicated by $\tilde{A}$ = (a, α, β). Thus $\tilde{A}$ = (a, α, β) ⊂ (a^L, a^U, α, β). T_rFN is a particular example of the general case T_pFN as a result as pictured in Figure 2.

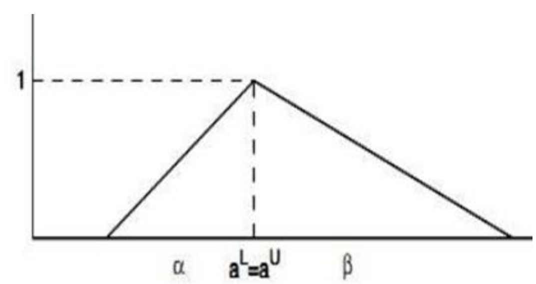

Figure 2: T_rFN

2.2. Fuzzy Algebra Properties of T_rFNs

As we prepare to formulate the Fuzzy Maximum Flow problem using T_rFNs, we introduce several arithmetic properties of these fuzzy numbers, which will serve as foundational elements for our approach.

Let $\tilde{A}_1, \tilde{A}_2 \in T_rFN$, such that $\tilde{A}_1 = (a_1, \alpha_1, \beta_1), \tilde{A}_2 = (a_2, \alpha_2, \beta_2)$. Based on (Hamadameen and Hassan, 2018a), the following rules apply:

1. Addition: $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1, \alpha_1, \beta_1) \oplus (a_2, \alpha_2, \beta_2) = (a_1 + a_2, \alpha_1 + \alpha_2,\ \beta_1 + \beta_2\ )$.
2. Image $\tilde{A}_1 = -\tilde{A}_1 = -(a_1, \alpha_1, \beta_1) = (-a_1, \beta_1, \alpha_1\ )$.
3. Subtraction: $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1, \alpha_1, \beta_1) \ominus (a_2, \alpha_2, \beta_2) = (a_1 - a_2, \alpha_1 + \beta_2,\ \beta_1 + \ \alpha_2)$.
4. Multiplication : $\tilde{A}_1 \otimes \tilde{A}_2 = (a_1, \alpha_1, \beta_1) \otimes (a_2, \alpha_2, \beta_2)$

$$\cong \begin{cases} (a_1 a_2, a_1 \alpha_2 + a_2 \alpha_1, a_1 \beta_2 + a_2 \beta_1;\ \tilde{A}_1, \tilde{A}_2 > \tilde{0} \\ (a_1 a_2, -a_2 \alpha_1 - a_1 \beta_2, -a_2 \beta_1 - a_1 \alpha_2;\ \tilde{A}_1 < \tilde{0}\,, \tilde{A}_2 > \tilde{0} \\ (a_1 a_2, -a_2 \beta_1 - a_1 \beta_2, -a_2 \alpha_1 - a_1 \alpha_2;\ \tilde{A}_1, \tilde{A}_2 < \tilde{0} \end{cases}$$

5. Scalar multiplication: $\rho \otimes \tilde{A}_2 = \rho \otimes (a_1, \alpha_1, \beta_1) = \begin{cases} (\rho a_1, \rho \alpha_1, \rho \beta_1); \rho > 0 \\ (\rho a_1, -\rho \beta_1, -\rho \alpha_1); \rho < 0 \end{cases}$
6. Inverse: $\tilde{A}_1^{\ -1} (a_1, \alpha_1, \beta_1)^{-1} = ({a_1}^{-1}, \beta_1\, {a_1}^{-2}, \alpha_1 {a_1}^{-2})$.
7. Division: $\tilde{A}_1 \oslash \tilde{A}_2 = \tilde{A}_1 \otimes (\tilde{A}_2)^{-1} = (\frac{a_1}{a_2}, \frac{\beta_2 a_1 + \alpha_1 a_2}{{a_2}^2}, \frac{\alpha_2 a_1 + \beta_1 a_2}{{a_2}^2});\ \tilde{A}_1, \tilde{A}_2 > \tilde{0}$. Note that similar formulas hold when $\tilde{A}_1$ and or $\tilde{A}_2$ are negative.
8. $\tilde{A}_1 = \tilde{0} \Leftrightarrow (a_1, \alpha_1, \beta_1) = (0, 0, 0)$.

2.3. Linear Ranking Function and Its Applications

A highly effective approach to transforming an FLP problem into its deterministic form and comparing fuzzy numbers is through the use of the linear ranking function(Maleki, 2003, Hamadameen and Hassan, 2018a). Since this study examines fuzzy maximum flow as a specific instance of the FLP problem within a fuzzy framework using the linear ranking function, this section introduces the definition of a linear ranking function along with its key properties.

Definition 4: Linear Ranking Function ($\Re$). is a map which transforms each fuzzy number into its corresponding real line, where a natural order exists, mathematically $\Re: \tilde{A} \to \mathbb{R}, \forall \tilde{A}$ and $\mathbb{R}$ is the set of all real numbers(Ameen, 2015, Hamadameen and Zainuddin, 2015). Furthermore, the $\Re$'s fuzzy algebra rules on the fuzzy numbers are as follows, for all $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4 \in T_rFN$, and $\rho \in \mathbb{R}$.

1. $\tilde{A}_1 \ \geq_{\Re} \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) \geq \Re(\tilde{A}_2)$.
2. $\tilde{A}_1 \ >_{\Re} \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) > \Re(\tilde{A}_2)$.
3. $\tilde{A}_1 \ =_{\Re} \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) = \Re(\tilde{A}_2)$.
4. $\tilde{A}_1 \ \leq_{\Re} \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) \leq \Re(\tilde{A}_2)$.
5. $\rho \tilde{A}_1 +_{\Re} \tilde{A}_2 \Leftrightarrow \rho \Re(\tilde{A}_1) + \Re(\tilde{A}_2)$.
6. $\tilde{A}_1 \ =_{\Re} \tilde{0} \Leftrightarrow \Re(\tilde{A}_1) = \Re(\tilde{0}) = 0$.
7. $\tilde{A}_1 \ \geq_{\Re} \tilde{A}_2 \Leftrightarrow \Re(\tilde{A}_1) - \Re(\tilde{A}_2) \geq_{\Re} 0 \Leftrightarrow -\tilde{A}_2 \ \geq_{\Re} - \tilde{A}_1$.
8. $(\tilde{A}_1 \ \geq_{\Re} \tilde{A}_2) \wedge (\tilde{A}_3 \ \geq_{\Re} \tilde{A}_4) \Rightarrow \tilde{A}_1 + \ \tilde{A}_3 \geq_{\Re} \tilde{A}_2 + \tilde{A}_4$.

Ultimately, the most suitable ranking function for our work is the one employed by previous studies (Mahdavi-Amiri and Nasseri, 2007, Hamadameen and Hassan, 2018a). Where its mathematical form is as:

$$\Re(A) = a + \frac{\beta - \alpha}{4} \qquad (1)$$

3. Fuzzy Maximum Flow Problems

The maximal flow problem, first introduced by Fulkerson and Dantzig (D. R. Fulkerson and G. B. Dantzig, 1955), was initially solved using a specialized simplex method designed for linear programming. This approach optimized network flow calculations, paving the way for further advancements in solving flow-related problems efficiently. Ford and Fulkerson (L. R. Ford and D. R. Fulkerson, 1956,) addressed the maximal flow problem by developing the augmenting path algorithm, a method that iteratively increases the flow in a network until the maximum possible flow is achieved. This approach significantly improved efficiency in solving network flow problems and laid the foundation for further advancements in optimization techniques. Various algorithms have been developed to solve crisp maximal flow problems, each designed to optimize network flow efficiently. These methods focus on determining the maximum possible flow through a network with precise, well-defined parameters, ensuring accurate results in applications such as transportation, communication networks, and logistics (Ahuja et al., 1993, Bazaraa et al., 2011).

In real-world scenarios, parameters such as cost, capacities, and demand in the maximal flow problem may not always be precisely defined. Factors like uncertainty, variability, and incomplete data can make it difficult to express these values as fixed quantities, requiring more flexible approaches to model and solve such problems effectively. Given the uncertainty in key variables of the maximum flow problem, such as cost, capacities, and demand, these values can be represented using fuzzy numbers(Zadeh, 1965). This approach accounts for imprecision and variability in real-world scenarios. When maximum flow problems incorporate fuzzy variables, they are referred to as fuzzy maximum flow problems, providing a more flexible framework for decision-making in uncertain environments.

3.1. Basic Definitions of Networks

Definition 1. Graph. A graph or network is characterized by two sets of elements: nodes and arcs. We establish a set, denoted as V, consisting of points or vertices. In this context, the vertices of the graph or network are also referred to as nodes (Trudeau, 2013, Winston, 2002).

Definition 2. Arc. An arc is defined by a sequential pair of vertices and indicates a potential pathway or direction of movement between them (Trudeau, 2013, Winston, 2002).

Definition 3. Chain. A chain is a sequence of arcs in which each arc shares precisely one vertex with the preceding arc (Winston, 2002).

Definition 4. Path. A path is a chain where the endpoint of each arc precisely matches the starting point of the following arc (Winston, 2002).

Definition 5. Cut. Select any subset of nodes, denoted as V', which includes the sink but excludes the source. The collection of arcs (i, j), where i is not in V' and j belongs to V', constitutes a cut for the network (Winston, 2002, Henning and Van Vuuren, 2022). Definition 6. Capacity. The capacity of a cut is determined by summing the capacities of all arcs included in the cut(Atamtürk, 2002).

3.2. Lemmas

The following lemmas highlight the relationship between cuts and fuzzy maximum flows, serving as an extension of the deterministic flow and cut within the network (Winston, 1991, Riley, 2002, Elias et al., 2003, Sapundzhi and Popstoilov, 2020).

Lemma 1. The fuzzy flow from the source to the sink in any fuzzy feasible flow does not exceed the fuzzy capacity of any cut.

Proof. Think about a randomly chosen division within a network, represented by a set of nodes V'. This set includes the sink but excludes the source. Let V represent all the remaining nodes in the network, excluding those explicitly specified earlier. Define $\tilde{x_{ij}}$ as the fuzzy flow along the arc (i, j) within the framework of a fuzzy feasible flow. Similarly, let $f^{\sim}$ represent the fuzzy flow moving from the source to the sink within this fuzzy feasible flow system.

Adding up the fuzzy flow balance equations gives a comprehensive representation of the fuzzy feasible flow throughout the network, ensuring that the total incoming flow aligns with the total outgoing flow at each node, except for the source and sink:

$$\begin{aligned} &fuzzy\ flow\ out\ of\ node\ i \\ &\qquad - fuzzy\ flow\ into\ node\ i \\ &\qquad \cong 0^{\sim} \qquad (2) \end{aligned}$$

By applying equation (2) across all nodes i in V, we observe that terms related to arcs (i, j), where both i and j belong to V will cancel out. As a result, the remaining expression simplifies, leading us to the refined formulation of the fuzzy flow balance within the network:

$$\sum_{\substack{i \in V \\ j \in V'}} \tilde{x_{ij}} - \sum_{\substack{i \in V' \\ j \in V}} \tilde{x_{ij}} = f^{\sim} \qquad (3)$$

At this stage, the first summation in equation (3) does not exceed the fuzzy capacity of the cut. Given that each $\tilde{x_{ij}}$ is nonnegative, it follows that the overall fuzzy feasible flow remains bounded within this capacity constraint, ensuring consistency within the fuzzy flow framework.

$$f^{\sim} \preccurlyeq fuzzy\ capacity\ of\ the\ cut \qquad (4)$$

Thus, with these logical steps and constraints established, the lemma is successfully validated, confirming the correctness of the fuzzy flow framework within the given network structure.

Lemma 2. If the sink cannot be labeled then:

$$\begin{gathered} Fuzzy\ capacity\ of\ cut \\ \cong current\ fuzzy\ flow\ from\ source\ to\ sink \end{gathered}$$

Proof. Let V' represent the set of unlabeled nodes, while V denotes the set of labeled nodes. Now, consider an arc (i,j) where i belongs to V and j is part of V'.

Thus, $x_{ij}^{\sim} = fuzzy\ capacity\ of\ arc\ (i,j)$ must be maintained. Otherwise, j could be labeled through a forward arc, which would contradict its classification within V'. Now, take into account an arc (i,j) where i belongs to V' and j is in V. In this case, $x_{ij}^{\sim} = 0^{\sim}$ must be maintained; otherwise, node i could be labeled through a backward arc, contradicting its membership in V'.

Now, equation (3) in Lemma 1 demonstrates that the existing fuzzy flow must adhere to the following requirement:

$$\begin{gathered} Fuzzy\ capacity\ of\ cut \cong \\ current\ fuzzy\ flow\ from\ source\ to\ sink. \end{gathered}$$

4. Mathematical Formulation of the Fuzzy Maximum Flow as a Fuzzy Linear Programming Model

This section introduces the fuzzy linear programming approach to solving fuzzy maximum flow problems. It involves assigning the value of $x_{ij}^{\sim}$ based on the fuzzy current flowing through each arc (i,j) within a network, represented as $G(V,E)$, where V denotes the network's vertices and E signifies the arcs connecting these vertices with fuzzy upper capacity $u_{ij}^{\sim}$. Let $f^{\sim}$ represent the maximum fuzzy flow within a network, spanning from the source node xo to the destination node si. The following mathematical formulations are applicable in fuzzy environments for determining the fuzzy maximum flow (Kumar and Kaur, 2011, Dimri and Ram, 2018).

$$\begin{gathered} Maximize\ f^{\sim} \\ s.t. \\ \sum_j x_{ij}^{\sim} \cong \sum_k x_{ki}^{\sim} \oplus f^{\sim}; i = s \\ \sum_j x_{ij}^{\sim} \cong \sum_k x_{ki}^{\sim}; \forall i \neq s,t \\ \sum_j x_{ij}^{\sim} \oplus f^{\sim} \cong \sum_k x_{ki}^{\sim}; i = t \\ x_{ij}^{\sim} \preccurlyeq u_{ij}^{\sim}; \forall\ (i,j) \in E \\ x_{ij}^{\sim} \succcurlyeq 0^{\sim} \end{gathered} \qquad (5)$$

Since the proposed method optimizes fuzzy maximum flow problems using the Ford-Fulkerson method and a linear ranking function, equation (5) can be reformulated as shown in (6) within the framework of the linear ranking function ($\mathfrak{R}$) as used in the literature(Ameen, 2015, Hamadameen and Hassan, 2018a, Hamadameen, 2017). Meanwhile, the application of the Ford-Fulkerson method will be discussed and elaborated in the next section.

$$\begin{gathered} Maximize\ f^{\sim} \\ s.t. \\ \sum_j x_{ij}^{\sim} \cong_{\mathfrak{R}} \sum_k x_{ki}^{\sim} \oplus f^{\sim}; i = s \\ \sum_j x_{ij}^{\sim} \cong_{\mathfrak{R}} \sum_k x_{ki}^{\sim}; \forall i \neq s,t \\ \sum_j x_{ij}^{\sim} \oplus f^{\sim} \cong_{\mathfrak{R}} \sum_k x_{ki}^{\sim}; i = t \\ x_{ij}^{\sim} \preccurlyeq_{\mathfrak{R}} u_{ij}^{\sim}; \forall\ (i,j) \in E \\ x_{ij}^{\sim} \succcurlyeq_{\mathfrak{R}} 0^{\sim} \end{gathered} \qquad (6)$$

5. Ford-Fulkerson Fuzzy Method for Solving Maximum Flow Problems

We are advancing scientific knowledge by enhancing the traditional Ford-Fulkerson method, transforming it into a more innovative and efficient approach for handling fuzziness networks. This adaptation, known as the Ford-Fulkerson fuzzy method, is designed to effectively solve maximum fuzzy flow problems.

We begin with the assumption that a fuzzy feasible flow has already been established. By setting the fuzzy feasible flow in each arc to $0^{\sim}$, we still obtain a valid fuzzy feasible flow. With this foundation in place, we now turn our focus to addressing the following key questions.

How can we determine whether a fuzzy feasible flow is truly optimal and maximizes $0^{\sim}$? And, how can we adjust a fuzzy feasible flow to transform it into a new fuzzy feasible flow with an increased fuzzy flow from the source to the sink?

By addressing the second question, we identify the specific fuzzy properties associated with each arc within the fuzzy network:

1. If the fuzzy flow through arc (i,j) is less than its fuzzy capacity, there is room for increasing the fuzzy flow. Therefore, we define I as the set of arcs exhibiting this characteristic.
2. The fuzzy flow in arc (i,j) is positive. In this case, the fuzzy flow through arc (i,j) can be reduced. For this reason, we let R be the set of arcs with this property.

We begin the Ford-Fulkerson fuzzy labeling procedure to methodically adjust the fuzzy flow within the network. This process involves defining the labeling approach used to modify a fuzzy feasible flow, with the goal of improving the fuzzy flow from the source to the sink. This method, referred to as the Fuzzy Solution Algorithm via Ford-Fulkerson Method and Linear Ranking Function (FSA-FFMLR), is structured as follows:

Step 1: Label the source.

Step 2: Rely on the following rules to label nodes and arcs except for arc $a_o^{\sim}$:

1. If node x already has a label, and node y does not, but the arc (x,y) belongs to set I, then; label node y, label arc (x,y), and arc (x,y) is considered a forward arc.
2. If node y is unlabeled, and node x is labeled, and the arc (y,x) belongs to set R, then; label node y, label arc (y,x), and arc (y,x) is classified as a backward arc.

Step 3: Keep applying this labeling procedure until either the sink is successfully labeled or no additional vertices remain that can be labeled. This ensures the process reaches a logical conclusion while maintaining the conditions necessary for fuzzy feasible flow.

Step 4: If the sink receives a label during the process, a sequence of labeled arcs referred to as C will form a path that connects the source to the sink. By modifying the fuzzy flow along the arcs in C, we can ensure that the flow remains feasible while also enhancing the total fuzzy flow from the source to the sink. To understand this, note that C must be composed of one of the following Scenarios:

First scenario: C is made up exclusively of forward arcs.

Second scenario: C contains both forward and backward arcs.

In both cases, it is possible to derive a new fuzzy feasible flow that surpasses the current one in terms of fuzzy flow from the source to the sink.

In the first case, the chain C is composed solely of forward arcs. For each forward arc within C, let $i^{\sim}(x,y)$ represent the fuzzy increment by which the fuzzy flow in arc (x,y) can be enhanced without breaching the fuzzy capacity constraint of that arc.

Let $k^{\sim} = min_{(x,y)\in C} i^{\sim}(x,y), k^{\sim} \succ 0^{\sim}$.

To establish a new fuzzy flow, increase the fuzzy flow in each arc of C by $k^{\sim}$ units. This adjustment does not breach any feasible capacity constraints and continues to uphold the conservation of fuzzy flow. As a result, the updated fuzzy flow remains feasible, effectively transporting $k^{\sim}$ additional units from source to sink compared to the existing fuzzy feasible flow.

In the second scenario, the chain C that connects the source to the sink includes both forward and backward arcs. For each balanced arc within C, let $r^{\sim}(x,y)$, denote the fuzzy quantity by which the fuzzy flow through arc (x,y) can be decreased.

Let $k_1^{\sim} = min_{(x,y)\in C\cap R} r^{\sim}(x,y)$ and $k_2^{\sim} = min_{(x,y)\in C\cap I} i^{\sim}(x,y), k_1^{\sim}, k_2^{\sim}, min\,(k_1^{\sim}, k_2^{\sim}) \succ 0^{\sim}$

Step 5: To enhance the fuzzy flow from source to sink while preserving a fuzzy feasible flow, reduce the fuzzy flow in all backward arcs of C by $min\,(k_1^{\sim}, k_2^{\sim})$, and simultaneously increase the fuzzy flow in all forward arcs of C by $min\,(k_1^{\sim}, k_2^{\sim})$. This adjustment maintains the conservation of fuzzy flow and ensures that no arc violates its fuzzy capacity constraints.

Since the final arc in C is a forward arc directed toward the sink, we have successfully identified a new fuzzy feasible flow. Consequently, the total fuzzy flow entering the sink has increased by $min\,(k_1^{\sim}, k_2^{\sim})$.

Step 6: Formulate a fuzzy network problem as a Fully Fuzzy Linear Programming (FFLP) problem within the Ford-Fulkerson method and linear ranking function.

Step 7: Use the fuzzy simplex method with a linear ranking function to identify the fuzzy maximum flow, which represents the fuzzy solution within the fuzzy network.

6. Illustrative Example

We present an example of applying the research methodology to the problem statement within the fuzzy maximum flow problem, utilizing the Ford-Fulkerson method and the linear ranking function framework to demonstrate the applicability of our mechanism. Additionally, we highlight the effectiveness of the innovative approach in optimizing the transport of large quantities of liquids and essential supplies from their original sources to designated locations through transportation networks.

The National Oil Company (NOC) aims to transport the maximum possible amount of oil per hour through a pipeline system from the exploration and extraction site (denoted as node so) to the final destination (denoted as node si), as illustrated in Figure 1. Along the route from so to si, the oil must pass through one or more of stations 1, 2, and 3. The various arcs in the network represent pipelines of different diameters. The fuzzy maximum number of barrels of oil (in millions per hour) that can be transported through each arc is provided in Table 1, where each fuzzy value represents a fuzzy arc capacity. Develop a Fuzzy Linear Programming (FLP) model to determine the fuzzy maximum number of barrels of oil per hour that can be transported from so to si.

Table 1: Fuzzy arc capacities for NOC

Fuzzy arc	Fuzzy capacity
$(so,1)$	$(3, \frac{1}{3}, \frac{1}{4})$
$(so,2)$	$(4, \frac{1}{2}, \frac{1}{4})$
$(1,2)$	$(4, \frac{1}{3}, \frac{1}{3})$
$(1,3)$	$(5, \frac{3}{4}, \frac{1}{4})$
$(3,si)$	$(2, \frac{1}{2}, \frac{1}{3})$
$(2,si)$	$(3, \frac{1}{3}, \frac{1}{4})$

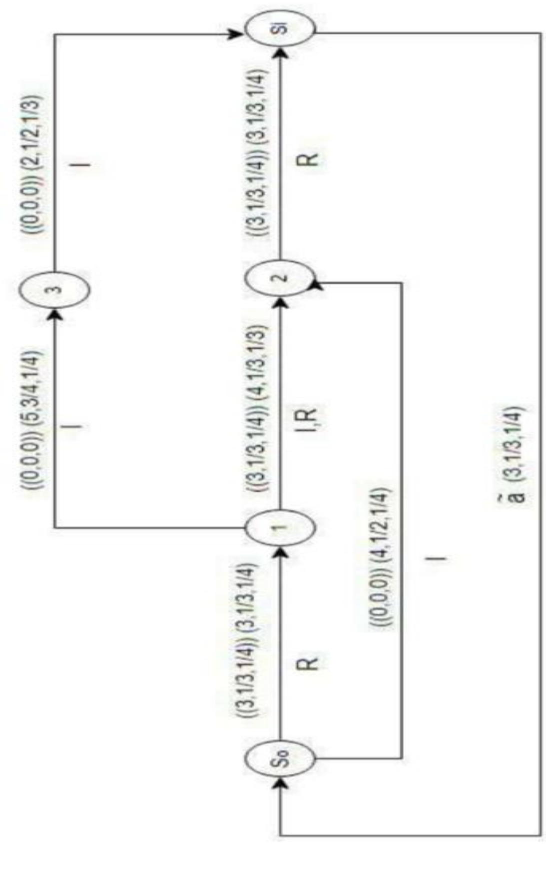

Figure 1: Fuzzy flow to sink= $(3,\frac{1}{3},\frac{1}{4})$ via the chain $(so,2)-(1,2)-(1,3)-(3,si)$

Solution: The node so is referred to as the source node because oil flows out of it fuzzily, albeit somewhat indistinctly, while no oil enters it.

Likewise, the node si is called the sink node because oil flows into it in a fuzzy manner, but none flows out. To clarify certain aspects, we have introduced an artificial arc, labeled ao, connecting the sink to the source. The indistinct flow through ao does not consist of actual oil, which is why it is referred to as an artificial arc.

To establish the Fuzzy Linear Programming (FLP) model that will produce the maximum fuzzy flow from node so to node si, we recognize that the National Oil Company (NOC) must decide the amount of oil (per hour) to be transmitted with uncertainty through arc (i,j). Consequently, we introduce the following definitions:

$$x^{\sim}{}_{ij} = fuzzy\ millions\ of\ barrels\ of\ oil\ per\ hour$$

$$that\ will\ pass\ through\ arc\ (i,j) of\ pipline \qquad (7)$$

A feasible fuzzy flow must satisfy two key characteristics:

$$0^{\sim} \preccurlyeq fuzzy\ flow\ through\ each\ arc \preccurlyeq arc\ fuzzy\ capacity \qquad (8)$$

Or, fuzzy capacity constraints have to satisfies; the fuzzy flow through each pipeline segment must not exceed its fuzzy maximum allowable capacity, accounting for uncertainty in measurements.

$$fuzzy\ flow\ into\ node\ i \cong fuzzy\ flow\ out\ of\ node\ i \qquad (9)$$

In other words; fuzzy flow conservation has to satisfies at each node (except for source and destination), the total incoming fuzzy flow must equal the total outgoing fuzzy flow, ensuring a balanced transfer.

Assuming no oil is lost during transportation through the network, a fuzzy feasible flow must meet the conservation of fuzzy flow constraint at each node in (3). The inclusion of the fuzzy artificial arc ao ensures that this constraint is also maintained at both the source and the sink.

Defining xo as the fuzzy flow passing through the artificial arc, the conservation of fuzzy flow principle ensures that the total inflow and outflow remain balanced across the network, particularly at the source and sink, that implies to:

$$x_o^{\sim} \cong total\ fuzzy\ amount\ of\ oil\ entering\ the\ sink \qquad (10)$$

The objective in (10) of NOC is to maximize $x_o^{\sim}$ subject to (7), (8) and (9) as follows in (10):

$$\begin{gathered} z^{\sim} \cong_{\Re} x_o^{\sim}; \\ \max\left[x_o^{\sim}\ is\ the\ fuzzy\ total\ amount\ of\ oil\ entering\ the\ sink\right] \\ s.t. \\ x_{so,1}^{\sim} \preccurlyeq_{\Re} \left(3, \frac{1}{3}, \frac{1}{4}\right) \\ x_{so,2}^{\sim} \preccurlyeq_{\Re} \left(4, \frac{1}{2}, \frac{1}{4}\right) \\ x_{1,2}^{\sim} \preccurlyeq_{\Re} \left(4, \frac{1}{3}, \frac{1}{3}\right) \\ x_{2,si}^{\sim} \preccurlyeq_{\Re} \left(3, \frac{1}{3}, \frac{1}{4}\right) \\ x_{1,3}^{\sim} \preccurlyeq_{\Re} \left(5, \frac{3}{4}, \frac{1}{4}\right) \\ x_{3,si}^{\sim} \preccurlyeq_{\Re} \left(2, \frac{1}{2}, \frac{1}{3}\right) \\ x_o^{\sim} \cong_{\Re} x_{so,1}^{\sim} \oplus x_{so,2}^{\sim} \\ x_{o,1}^{\sim} \cong_{\Re} x_{1,2}^{\sim} \oplus x_{1,3}^{\sim} \\ x_{so,2}^{\sim} \oplus x_{1,2}^{\sim} \cong_{\Re} x_{2,si}^{\sim} \\ x_{1,3}^{\sim} \cong_{\Re} x_{3,si}^{\sim} \\ x_{3,si}^{\sim} \oplus x_{2,si}^{\sim} \cong_{\Re} x_o^{\sim} \\ x_{i,j}^{\sim} \succcurlyeq_{\Re} 0^{\sim} \end{gathered} \qquad (11)$$

Equation (11) is expressed in its standard form as shown in (12).

$$
\begin{gathered}
max\, z^{\sim} \cong_{\Re} \\
(1,0,0)x_o^{\sim} \oplus (0,0,0)x_{so,1}^{\sim} \oplus \\
(0,0,0)x_{so,2}^{\sim} \oplus (0,0,0)x_{1,2}^{\sim} \oplus \\
(0,0,0)x_{2.si}^{\sim} \oplus (0,0,0)x_{1,3}^{\sim} \oplus \\
(0,0,0)x_{3,si}^{\sim} \oplus \\
(0,0,0)\sum_{k=1}^{6} s_k^{\sim} \ominus \\
M^{\sim}\sum_{l=1}^{5} a_l^{\sim} \\
s.t. \\
x_{so,1}^{\sim} \oplus s_1^{\sim} \cong_{\Re} \left(3,\frac{1}{3},\frac{1}{4}\right) \\
x_{so,2}^{\sim} \oplus s_2^{\sim} \cong_{\Re} \left(4,\frac{1}{2},\frac{1}{4}\right) \\
x_{1,2}^{\sim} \oplus s_3^{\sim} \cong_{\Re} \left(4,\frac{1}{3},\frac{1}{3}\right) \\
x_{2,si}^{\sim} \oplus s_4^{\sim} \cong_{\Re} \left(3,\frac{1}{3},\frac{1}{4}\right) \qquad (12) \\
x_{1,3}^{\sim} \oplus s_5^{\sim} \cong_{\Re} \left(5,\frac{3}{4},\frac{1}{4}\right) \\
x_{3,si}^{\sim} \oplus s_6^{\sim} \cong_{\Re} \left(2,\frac{1}{2},\frac{1}{3}\right) \\
x_o^{\sim} \ominus x_{so,1}^{\sim} \ominus x_{so,2}^{\sim} \oplus a_1^{\sim} \\
\cong_{\Re} (0,0,0) \\
x_{so,1}^{\sim} \ominus x_{1,2}^{\sim} \ominus x_{1,3}^{\sim} \oplus a_2^{\sim} \\
\cong_{\Re} (0,0,0) \\
x_{so,2}^{\sim} \oplus x_{1,2}^{\sim} \ominus x_{2,si}^{\sim} \oplus a_3^{\sim} \\
\cong_{\Re} (0,0,0) \\
x_{1,3}^{\sim} \ominus x_{3,si}^{\sim} \oplus a_4^{\sim} \\
\cong_{\Re} (0,0,0) \\
x_{3,si}^{\sim} \oplus x_{2,si}^{\sim} \ominus x_o^{\sim} \oplus a_5^{\sim} \\
\cong_{\Re} (0,0,0) \\
x_{i,j}^{\sim}, s_k^{\sim}, a_l^{\sim} \succcurlyeq_{\Re} 0^{\sim} \\
;\ M^{\sim}\ is\ very\ big\ fuzzy\ number
\end{gathered}
$$

Applying Hamadamen's method (Hamadameen, 2017) to solve equation (12) yields the following fuzzy optimal result;

$$
\begin{aligned}
&Fuzzy\ optimal\ solution \\
&= \{x_o^{\sim}; x_{so,1}^{\sim}, x_{so,2}^{\sim}, x_{1,2}^{\sim}, x_{2.si}^{\sim}, x_{1,3}^{\sim}, x_{3,si}^{\sim}\} \\
&= \{\left(5,\frac{5}{6},\frac{7}{12}\right); \left(2,\frac{1}{2},\frac{1}{3}\right), \left(3,\frac{1}{3},\frac{1}{4}\right)
\end{aligned}
$$

$, (0,0,0), \left(3,\frac{1}{3},\frac{1}{4}\right), \left(2,\frac{1}{2},\frac{1}{3}\right), (2,\frac{1}{2},\frac{1}{3})\}.$

The optimal fuzzy oil flow from node so to node si reaches $(5,\frac{5}{6},\frac{7}{12})$ million barrels per hour. This flow is distributed among three distinct routes, with each carrying $(2,\frac{1}{2},\frac{1}{3})$ million barrels per hour: $so - 1 - 2 - si, so - 1 - 3 - si$, and $so - 2 - si$, as illustrated in Figure 2. This serves as the fuzzy optimal solution for problem (11), meeting the fuzzy feasibility conditions within the framework of the linear ranking function in (1), (2), (3), and (4). Furthermore, its properties facilitate the comparison of fuzzy numbers, as described in Section II, while incorporating the environment of Lemma 1 and Lemma 2 in Section III, with verification provided below.

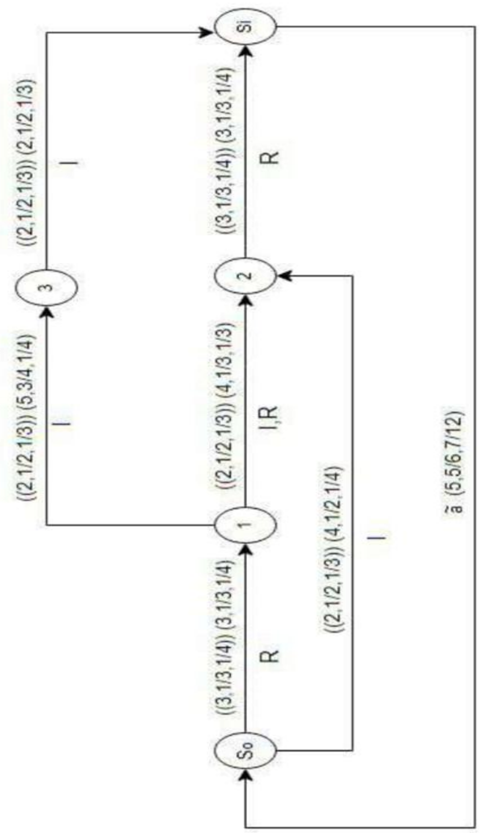

Figure 2: Maximum fuzzy flow from source to sink= $(5,\frac{5}{6},\frac{7}{12})$

$$\left(2,\frac{1}{2},\frac{1}{3}\right) \preccurlyeq_{\mathfrak{R}} \left(3,\frac{1}{3},\frac{1}{4}\right)$$
$$\Rightarrow 1\frac{23}{24} \leq 2\frac{47}{48}$$
$$\left(3,\frac{1}{3},\frac{1}{4}\right) \preccurlyeq_{\mathfrak{R}} \left(4,\frac{1}{2},\frac{1}{4}\right)$$
$$\Rightarrow 2\frac{47}{48} \leq 3\frac{15}{16}$$
$$(0,0,0) \preccurlyeq_{\mathfrak{R}} \left(4,\frac{1}{3},\frac{1}{3}\right)$$
$$\Rightarrow 0 \leq 4$$
$$\left(3,\frac{1}{3},\frac{1}{4}\right) \preccurlyeq_{\mathfrak{R}} \left(3,\frac{1}{3},\frac{1}{4}\right)$$
$$\Rightarrow 2\frac{47}{48} = 2\frac{47}{48}$$
$$\left(2,\frac{1}{2},\frac{1}{3}\right) \preccurlyeq_{\mathfrak{R}} \left(3,\frac{1}{3},\frac{1}{4}\right) \qquad (13)$$
$$\Rightarrow 1\frac{23}{24} \leq 2\frac{47}{48}$$
$$\left(2,\frac{1}{2},\frac{1}{3}\right) \preccurlyeq_{\mathfrak{R}} \left(5,\frac{3}{4},\frac{1}{4}\right)$$
$$\Rightarrow 1\frac{23}{24} \leq 4\frac{7}{8}$$
$$(5,\frac{5}{6},\frac{7}{12}) \cong_{\mathfrak{R}} \left(2,\frac{1}{2},\frac{1}{3}\right) \oplus \left(3,\frac{1}{3},\frac{1}{4}\right)$$
$$\Rightarrow \left(5,\frac{5}{6},\frac{7}{12}\right) \cong_{\mathfrak{R}} \left(5,\frac{5}{6},\frac{7}{12}\right)$$
$$\Rightarrow 4\frac{15}{16} = 4\frac{15}{16}$$
$$\left(2,\frac{1}{2},\frac{1}{3}\right) \cong_{\mathfrak{R}} (0,0,0) \oplus \left(2,\frac{1}{2},\frac{1}{3}\right)$$
$$\Rightarrow 1\frac{23}{24} = 1\frac{23}{24}$$
$$(3,\frac{1}{3},\frac{1}{4}) \oplus (0,0,0) \cong_{\mathfrak{R}} \left(3,\frac{1}{3},\frac{1}{4}\right)$$
$$\Rightarrow 2\frac{47}{48} = 2\frac{47}{48}$$
$$(2,\frac{1}{2},\frac{1}{3}) \cong_{\mathfrak{R}} (2,\frac{1}{2},\frac{1}{3})$$
$$\Rightarrow 1\frac{23}{24} = 1\frac{23}{24}$$
$$\left(2,\frac{1}{2},\frac{1}{3}\right) \oplus \left(3,\frac{1}{3},\frac{1}{4}\right) \cong_{\mathfrak{R}} (5,\frac{5}{6},\frac{7}{12})$$
$$\Rightarrow \left(5,\frac{5}{6},\frac{7}{12}\right) = (5,\frac{5}{6},\frac{7}{12})$$
$$\Rightarrow 4\frac{15}{16} = 4\frac{15}{16}$$

7.A Systematic Justification of the Proposed Approach to Fuzzy Maximum Flow

This section presents a logical analysis of the method introduced in this study. By examining the network's illustrative Figure 2 and considering the obtained results, we arrive at the following explanation.

In the figure, (so,1) $\in R; (so,2) \in I; (1,3) \in I; (1,2) \in I and\ R; (2,si) \in R; (3,si) \in I$. We start by labeling arc $(so,2)$ and node 2, as arc $(so,2)$ is a forward arc. Next, we label arc $(1,2)$ and node 1. Arc $(1,2)$ is considered a backward arc because node 1 was unlabeled before we marked arc $(1,2)$, and this arc belongs to set R. Since nodes so, 1, and 2 have been labeled, we proceed by labeling arc $(1,3)$ and node 3, since arc $(1,3)$ is a forward arc and node 3 remains unlabeled. Finally, arc $(3,si)$ and node si has not yet been labeled, as arc $(3,si)$ is also a forward arc and node si has yet to be labeled. At this stage, the sink si has been labeled through the chain $C = (so,2) - (1,2) - (1,3) - (3,si)$. Except for arc $(1,2)$, all arcs within the chain are forward arcs. The values associated with these arcs and nodes are:

$i(so,2) = \left(4,\frac{1}{2},\frac{1}{4}\right);\ i(1,3) = \left(5,\frac{3}{4},\frac{1}{4}\right);\ i(3,si) = \left(2,\frac{1}{2},\frac{1}{3}\right);\ and\ r(1,2) = (4,\frac{1}{3},\frac{1}{3})$.

Because,

$\min \mathfrak{R}\left\{\left(4,\frac{1}{2},\frac{1}{4}\right),\left(5,\frac{3}{4},\frac{1}{4}\right),\left(2,\frac{1}{2},\frac{1}{3}\right)\right\} = \min\left\{3\frac{15}{16},4\frac{7}{8},1\frac{23}{24}\right\} = 1\frac{23}{24}$. $\min \mathfrak{R}\left\{\left(4,\frac{1}{3},\frac{1}{3}\right)\right\} = 4$.

Therefore,

$min_{(x,y)\in C\cap I}\, i(x,y) = \left(2,\frac{1}{2},\frac{1}{3}\right) \wedge min_{(x,y)\in C\cap R}\, r(x,y) = \left(4,\frac{1}{3},\frac{1}{3}\right) = \left(2,\frac{1}{2},\frac{1}{3}\right)$.

By adjusting the fuzzy flow in network C, we can increase the flow on all forward arcs by $(2,\frac{1}{2},\frac{1}{3})$ and decrease it on all backward arcs by the same amount, as illustrated in Figure 2. This adjustment results in an increased fuzzy flow from the source to the sink by $(2,\frac{1}{2},\frac{1}{3})$, specifically between nodes 2 and 3. To achieve this, we reroute $(2,\frac{1}{2},\frac{1}{3})$ units previously transported via arc $(1,2)$ to the alternative path $1-3-si$. This redirection allows an additional $(2, \frac{1}{2}, \frac{1}{3})$ units to be transported from the source to the sink through the path $so - 2 - si$. Notably, the idea of a backward arc plays a crucial role in identifying this improved fuzzy flow. Moreover, if the sink cannot be labeled, the current fuzzy flow is considered optimal, as demonstrated in Lemma 2.

8.Advantages of the Proposed Method for Addressing Network Uncertainty

Optimizing fuzzy maximum flow problems using the Ford-Fulkerson method combined with a linear ranking function offers several advantages, particularly in handling uncertainty within network flow optimization. Here are some key benefits:

1. Improved Handling of Uncertainty – Fuzzy logic allows for better representation of uncertain or imprecise data, making the method more applicable to real-world scenarios(Bozhenyuk and Gerasimenko, 2013, Jiang et al., 2013).
2. Enhanced Computational Efficiency – The Ford-Fulkerson algorithm is known for its efficiency in solving maximum flow problems, and integrating a ranking function can further streamline calculations (Jiang et al., 2013, Kyi and Naing, 2018).
3. Better Decision-Making – The linear ranking function helps prioritize flow paths based on fuzzy parameters, leading to more informed and optimized routing decisions (Kumar and Kaur, 2011, Kyi and Naing, 2018, Sapundzhi and Popstoilov, 2020).
4. Flexibility in Network Applications – This approach is useful in various fields, including transportation, logistics, and communication networks, where flow optimization under uncertainty is crucial (Bozhenyuk and Gerasimenko, 2013).(Hernandes et al., 2007)
5. Robustness in Complex Systems – By incorporating fuzzy logic, the method can adapt to dynamic changes in network conditions, making it more resilient to fluctuations (Kumar and Kaur, 2011, Marpaung et al., 2023, Wahyuningsih et al., 2024).

9.Conclusions

In this paper, the Fuzzy Maximum Flow (FMF) problems have been defined in the Fully Fuzzy Networks (FFN) as a Fully Fuzzy Linear Programming (FFLP) problems, where the coefficients of the objective functions, constraints, right hand side parameters, and variables are of the type of the Triangular Fuzzy Number (T_rFNs). A solution strategy of such FFLP was presented in the circumstance of certain linear ranking function, namely, Fuzzy Solution Algorithm via Ford-Fulkerson Method and Linear Ranking Function (FSA-FFMLR). The revised simplex method together with Gaussian elimination in the environment of the linear ranking function which was described in the proposed was first converted from FFLP problem to partially SFFLP problem, and then was solved to find a fuzzy solution. The results show that the proposed FSA-FFMLR is applicable and practicable within computational applications. We intend to expand this research further to practical contributions. The work will be able to help solve real life and industrial problems which are usually complicated, uncertain and continuously subject to changes, by considering the fuzziness in the formulation of the model. In addition, potential research will be the application of our proposed solution procedure to real life problems which usually involve many variables and require quick optimum solutions. We believe that this study will spur other research so as to have positive impacts on organizational productivity and competitiveness in many industrial networks.

Acknowledgment: The author extends heartfelt thanks to the ZJPAS reviewers for their valuable feedback, which significantly enhanced both the substance and clarity of the paper.

Financial support: No financial support.

Potential conflicts of interest. The author declares no conflict of interest in relation to this article.

References

AHUJA, R. K., MAGNANTI, T. L. & ORLIN, J. B. 1993. Network flows: theory, algorithms, and applications, Prentice hall Englewood Cliffs, NJ.

AKRAM, M., HABIB, A. & ALLAHVIRANLOO, T. 2022. A new maximal flow algorithm for solving optimization problems with linguistic capacities and flows. Information Sciences, 612, 201-230.

AMEEN, A. O. H. 2015. Improved Two-phase Solution Strategy for Multiobjective Fuzzy Stochastic Linear Programming Problems with Uncertain Probability Distribution. Universiti Teknologi Malaysia.

ANTTALAINEN, T. 2003. Introduction to telecommunications network engineering, Artech House.

ATAMTÜRK, A. 2002. On capacitated network design cut–set polyhedra. Mathematical Programming, 92, 425-437.

BAST, H., DELLING, D., GOLDBERG, A., MÜLLER-HANNEMANN, M., PAJOR, T., SANDERS, P., WAGNER, D. & WERNECK, R. F. 2016. Route planning in transportation networks. Algorithm engineering: Selected results and surveys, 19-80.

BAZARAA, M. S., JARVIS, J. J. & SHERALI, H. D. 2011. Linear programming and network flows, John Wiley & Sons.

BOZHENYUK, A. & GERASIMENKO, E. 2013. Methods for maximum and minimum cost flow determining in fuzzy conditions. World Appl Sci J, 22, 76-81.

BOZHENYUK, A., GERASIMENKO, E. & ROZENBERG, I. The methods of maximum Flow and minimum cost flow finding in fuzzy network. Concept Discovery in Unstructured Data Workshop (CDUD 2012) co-located with the 10th International Conference on Formal Concept Analysis (ICFCA 2012) May, 2012. Citeseer, 1-12.

BOZHENYUK, A. V., GERASIMENKO, E. M., KACPRZYK, J. & ROZENBERG, I. N. 2017. Flows in networks under fuzzy conditions, Springer.

CHANAS, S., DELGADO, M., VERDEGAY, J. & VILA, M. 1995. Fuzzy optimal flow on imprecise structures. European Journal of Operational Research, 83, 568-580.

CHANAS, S. & KOLODZIEJCZYK, W. 1986. Integer flows in network with fuzzy capacity constraints. Networks, 16, 17-31.

CHANAS, S. & KOŁODZIEJCZYK, W. 1982. Maximum flow in a network with fuzzy arc capacities. Fuzzy Sets and Systems, 8, 165-173.

CHANAS, S. & KOŁODZIEJCZYK, W. 1984. Real-valued flows in a network with fuzzy arc capacities. Fuzzy Sets and Systems, 13, 139-151.

CHENG, Y., FAROOQ, S. & JOHANSEN, J. 2011. Manufacturing network evolution: a manufacturing plant perspective. International Journal of Operations & Production Management, 31, 1311-1331.

CHENG, Y., FAROOQ, S. & JOHANSEN, J. 2015. International manufacturing network: past, present, and future. International Journal of Operations & Production Management, 35, 392-429.

D. R. FULKERSON AND G. B. DANTZIG 1955. Computation of maximum flow in network. Naval Research Logisics Quarterly, 2, 277-283.

DAVIES, S., MOSELEY, B., VASSILVITSKII, S. & WANG, Y. Predictive flows for faster ford-fulkerson. International Conference on Machine Learning, 2023. PMLR, 7231-7248.

DELGADO, M., VERDEGAY, J. & VILA, M. 1985. On fuzzy tree definition. European Journal of Operational Research, 22, 243-249.

DIMRI, S. C. & RAM, M. 2018. Flow Maximization Problem as Linear Programming Problem with Capacity Constraints. Applications and Applied Mathematics: An International Journal (AAM), 13, 33.

DUBOIS, D. & PRADE, H. 1978. Operations on fuzzy numbers. International Journal of systems science, 9, 613-626.

EKANAYAKE, E., DAUNDASEKARA, W. & PERERA, S. 2022. New approach to obtain the maximum flow in a network and optimal solution for the transportation problems. Modern Applied Science, 16, 30.

ELIAS, P., FEINSTEIN, A. & SHANNON, C. 2003. A note on the maximum flow through a network. IRE Transactions on Information Theory, 2, 117-119.

FITZEK, F. H., GRANELLI, F. & SEELING, P. 2020. Computing in communication networks: from theory to practice, Academic Press.

GANDAL, N., SALANT, D. & WAVERMAN, L. 2003. Standards in wireless telephone networks. Telecommunications Policy, 27, 325-332.

GANESAN, K. & VEERAMANI, P. 2006. Fuzzy linear programs with trapezoidal fuzzy numbers. Annals of operations research, 143, 305-315.

GOLDBERG, A. V. & TARJAN, R. E. 1988. A new approach to the maximum-flow problem. Journal of the ACM (JACM), 35, 921-940.

HAMADAMEEN, A. O. 2017. A novel technique for solving multiobjective fuzzy linear programming problems. ARO-the scientific journal of Koya university, 5, 1-8.

HAMADAMEEN, A. O. Multiobjective Stochastic Programing: The Case of Production of Cakes. 2018 International Conference on Pure and Applied Science, 2018.

HAMADAMEEN, A. O. & HASSAN, N. 2018a. A compromise solution for the fully fuzzy multiobjective linear programming problems. Ieee Access, 6, 43696-43711.

HAMADAMEEN, A. O. & HASSAN, N. 2018b. Pareto optimal solution for multiobjective stochastic linear programming problems with partial uncertainty. International Journal of Mathematics in Operational Research, 12, 139-166.

HAMADAMEEN, A. O. & ZAINUDDIN, Z. M. 2015. A reciprocated result using an approach of multiobjective stochastic linear programming models with partial uncertainty. International Journal of Mathematics in Operational Research, 7, 395-414.

HENNING, M. A. & VAN VUUREN, J. H. 2022. Graph and Network Theory: An Applied Approach Using Mathematica®, Springer Nature.

HERNANDES, F., LAMATA, M. T., TAKAHASHI, M. T., YAMAKAMI, A. & VERDEGAY, J. L. An algorithm for the fuzzy maximum flow problem. 2007 IEEE International Fuzzy Systems Conference, 2007. IEEE, 1-6.

IBRAHIM, I. Y. & ZEBARI, I. M. 2025. A Comprehensive Review of the Ford-fulkerson Algorithm for Network Flow Problems. Asian Journal of Research in Computer Science, 18, 335-344.

JIANG, Z., HU, X. & GAO, S. A parallel ford-fulkerson algorithm for maximum flow problem. Proceedings of the International Conference on Parallel and Distributed Processing Techniques and Applications (PDPTA), 2013. The Steering Committee of The World Congress in Computer Science, Computer …, 70.

KADHIM, A. H., & AL-DABBAGH, M. A. 2025. Fuzzy Optimization of Flow Networks Using Hybrid Ranking Techniques. Journal of Intelligent Systems and Applications, 11, 3176-3187. .

KAUR, A. K. M. 2010. An algorithm for solving fuzzy maximal flow problems using generalized trapezoidal fuzzy numbers. International Journal of Applied Science and Engineering, 8, 109-118.

KIM, K. H. & ROUSH, F. W. 1982. Fuzzy flows on networks. Fuzzy Sets and Systems, 8, 35-38.

KUMAR, A., BHATIA, N. & KAUR, M. A new approach for solving fuzzy maximal flow problems. Rough Sets, Fuzzy Sets, Data Mining and Granular Computing: 12th International Conference, RSFDGrC 2009, Delhi, India, December 15-18, 2009. Proceedings 12, 2009. Springer, 278-286.

KUMAR, A. & KAUR, M. 2011. Solution of fuzzy maximal flow problems using fuzzy linear programming. A)= a, 2, 4.

KUMAR, A. & KAUR, M. 2012. An Improved Algorithmfor Solving Fuzzy Maximal Flow Problems. International Journal of Applied Science and Engineering, 10, 19-27.

KYI, M. T. & NAING, L. L. 2018. Application of Ford-Fulkerson algorithm to maximum flow in water distribution pipeline network. International Journal of Scientific and Research Publications, 8, 306-310.

L. R. FORD AND D. R. FULKERSON 1956, . Maximal flow through a network, . Canadian Journal of Mathematics, . 8, , 399-404.

LI, B., SPRINGER, J., BEBIS, G. & GUNES, M. H. 2013. A survey of network flow applications. Journal of Network and Computer Applications, 36, 567-581.

MAHDAVI-AMIRI, N. & NASSERI, S. H. 2006. Duality in fuzzy number linear programming by use of a certain linear ranking function. Applied mathematics and computation, 180, 206-216.

MAHDAVI-AMIRI, N. & NASSERI, S. H. 2007. Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables. Fuzzy sets and systems, 158, 1961-1978.

MAJUMDER, S., SAHA, B., ANAND, P., KAR, S. & PAL, T. 2018. Uncertainty based genetic algorithm with varying population for random fuzzy maximum flow problem. Expert Systems, 35, e12264.

MALEKI, H. 2003. Ranking functions and their applications to fuzzy linear programming.

MARPAUNG, F., ARNITA, A. & SARI, N. 2023. Maximal Flow of Transportation Network in Medan City Using Ford-Fulkerson Algorithm. International Journal of Science, Technology & Management, 4, 100-106.

RILEY, K. 2002. Introduction to mathematical programming, applications and algorithms. Mathematics and Computer Education, 36, 85.

SAPUNDZHI, F. & POPSTOILOV, M. 2020. Maximum-flow problem in networking. Bulgarian Chemical Communications, 52, 192-196.

SIMPSON, R. 2013. Lighting control: technology and applications, Routledge.

TRUDEAU, R. J. 2013. Introduction to graph theory, Courier Corporation.

WAHYUNINGSIH, S., OCTOVIANA, L. T. & QOHAR, A. Performance of the maximum flow algorithm and its application to traffic density. AIP Conference Proceedings, 2024. AIP Publishing.

WINSTON, W. 2002. Introduction to Mathematical Programming: Applications.

WINSTON, W. L. 1991. Introduction to Mathematical Programming: Applications and Algorithms, PWS-Kent Publishing Company.

ZADEH, L. A. 1965. Fuzzy sets. Information and control, 8, 338-353.

ZADEH, L. A. 1996. On fuzzy algorithms. fuzzy sets, fuzzy logic, and fuzzy systems: selected papers By Lotfi A Zadeh. World Scientific.