Journal of Iraqi Al-Khwarizmi (JIKh) Volume:10 Issue:2 Year: 2026 pages: 155-172

Derivation Error Formulas for the *SySy* Technique and a Study of its Efficiency in Calculating Double Integrals Improper and Improper Derivatives Numerically

Authors Names	**ABSTRACT**
[a]*Yusra Gateea Abbood Al-Ameri* [b]*Safaa M. Aljassas* *Publication data: 7 /8 /2026* *Keywords: Double integrals, continuous integrand, improper derivatives, Romberg's acceleration.*	The primary objective of this study is to derive a general formula for the correction terms associated with the proposed $SySy$ method, particularly in cases where the integrand is continuous while its derivatives are improper, or when the integrand exhibits impropriety at one or more points within the domain of integration. This study presents several theorems aimed at determining error expressions for such types of double integrals, whether the impropriety arises at the lower bounds, upper bounds, or both simultaneously. Five illustrative examples are provided to demonstrate various cases of double integrals, and supplementary tables are included to present detailed results.

1. Introduction

Double integrals play a crucial role in determining surface areas, as well as in evaluating centroids and moments of inertia for plane surfaces, in addition to computing the volume beneath a surface represented by a double integral. For example, the volume generated by rotating the heart-shaped curve $\rho = 2(1 - \cos\theta)$about the polar axis can be obtained using double integrals. Furthermore, as stated by Frank Eyers in [2], the significance of double integrals also lies in calculating the area of a portion of the sphere $\psi^2 + ð^2 + \#^2 = 36$contained within the cylinder $ð^2 + \#^2 = 6ð$.

Numerous studies have addressed double integrals. In 1973, Schjear-Jacobsen [7] emphasized the computation of double integrals with continuous integrands of the form $\beta(\psi, ð) = \beta_1(\psi)\beta_2(ð)$, whereas Davies and Rabinowitz [9] in 1975 investigated integrals involving improper integrands.

In 2009, Dheyaa [3] proposed four numerical methods based on Romberg's acceleration combined with the midpoint and Simpson methods, namely $RM(RS), RS\,(RS), RS(RM)$ and $RM(RM)$ for evaluating double integrals with continuous integrands that are either improper or possess improper derivatives. Among these methods, $RM(RM)$ was identified as the most effective in terms of accuracy and convergence speed toward the exact values of the integrals.

In 2011, [6] examined three numerical methods that integrate Romberg's acceleration with two Newton–Cotes formulations (Simpson and Midpoint), referred to as RSS , RMS , and RSM, under the condition that the number of subintervals along both the X and Y axes is equal. These methods were applied to evaluate double integrals whose integrands have improper derivatives or are improper. The RSS method demonstrated superior performance in terms of accuracy and convergence speed, particularly when using a limited number of subintervals. Additional details can be found in [1,4]. In this study, a general formulation for the correction terms of the proposed $SySy$ method, introduced in [5], has been derived for cases where the integrand is continuous with improper derivatives or is improper at one or more points within the integration domain.

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2. Suggested technique (Sy)

The Suggested that represented by Safaa and Yusra derived from the Newton-Cotes formulas which are used to calculate the single integrals and gave good results in accuracy and speed of approach. The general formula of this technique is:

$$Sy\,(\gamma) = \frac{\gamma}{6}\Big(5\beta(a) + 5\beta(b) + 6\sum_{i=1}^{n} \beta(a + (i - 0.5)\gamma) + 10\sum_{i=1}^{\frac{n}{1}-1} \beta(a + 2i\gamma) + 14\sum_{i=1}^{\frac{n}{2}} \beta(a + (2i - 1)\gamma)$$

the correction terms series in the case of a continuous integral function are:

$$E_{Sy}(\gamma) = \frac{1}{12}\gamma^2(\beta'_{2n} - \beta'_0) - \frac{17}{720}\gamma^4\left(\beta_{2n}^{(3)} - \beta_0^{(3)}\right) + \frac{81}{30240}\gamma^6\left(\beta_{2n}^{(5)} - \beta_0^{(5)}\right) + \cdots$$

And the correction terms series in the case of improper integral function or derivative in lower end are:

$$E_{sy}(\gamma) = \left[-\frac{1}{36}\gamma^2 D + \frac{1}{36}\gamma^3 D^2 - \frac{13}{360}\gamma^4 D^3 + \cdots \right]\beta_1 + A_{Sy}\gamma^2 + B_{Sy}\gamma^4 + \cdots$$

the correction terms series in the case of improper integral function or derivative in upper end are:

$$E_{sy}(\gamma) = \left[\frac{1}{36}\gamma^2 D + \frac{1}{36}\gamma^3 D^2 + \frac{13}{360}\gamma^4 D^3 - \cdots \right]\beta_{2n} + A_{Sy}\gamma^2 + B_{Sy}\gamma^4 + \cdots$$

3. Derivation the error formula using the suggested technique ***SYSY* to calculate the double integral with improper derivative from the lower end:**

Theorem (1): Let the function $\beta(\psi, ð)$ be continuous and differentiable at each point of the integration region except at the point$(\psi, ð) = (\psi_0, ð_0)$, the approximate value of the double integral $J = \int_{ð_0}^{ð_n}\int_{\psi_0}^{\psi_n} f(\psi, ð)\,d\psi\,dð$ can be evaluated using the following $SYSY$ technique

And the error formula is :

$$\Big[E = -R_1\gamma^6\left(D_\psi^{(4)} + D_ð^{(4)}\right)$$
$$- R_2\gamma^8(R_3\left(D_\psi^{(6)} R_3 D_ð^{(6)} - R_4 D_\psi^{(5)} D_ð - R_4 D_\psi D_ð^{(5)} - R_5 D_\psi^{(4)} D_ð^{(2)} - R_5 D_\psi^{(2)} D_ð^{(4)}\right)$$
$$+ R_2 h^9\left(-R_3 D_\psi^{(7)} - R_3 D_ð^{(7)} + \cdots\right) + \cdots\Big]\,\beta(\psi_1, ð_1) + \lambda_{\,sy}\gamma^2 + \epsilon_{sy}\,\gamma^4 + \theta_{sy}\gamma^6 + \cdots$$

Where λ_{sy} , ϵ_{sy} , θ_{sy} +, ... are constants that depend on the partial derivatives of the function.

Proof: Suppose that the function $f(\psi, ð)$ is defined at each point of the integration region $[\psi_0, \psi_n] \times [ð_0, ð_n]$ and it is not improper and partial derivatives of the function are not defined at the point$(\psi_0, ð_0)$. This means that the Taylor's series of two-variable functions exists at each point of the integration region except at $(\psi_0, ð_0)$, [7].

We can write the double integral J by :

$$J = \int_{ð_0}^{ð_n}\int_{\psi_0}^{\psi_n}\beta(\psi,ð)d\psi dð$$
$$= \int_{ð_0}^{ð_1}\int_{\psi_0}^{\psi_1}\beta(\psi,ð)d\psi dð + \int_{ð_0}^{ð_1}\sum_{r=1}^{n-1}\int_{\psi_r}^{\psi_{r+1}}\beta(\psi,ð)d\psi dð$$
$$+ \sum_{s=1}^{n-1}\int_{ð_s}^{ð_{s+1}}\int_{\psi_0}^{\psi_1}\beta(\psi,ð)d\psi dð \int_{ð_1}^{ð_n}\int_{\psi_1}^{\psi_n}\beta(\psi,ð)d\psi dð \quad \ldots \quad (1)$$

For the first integral on the partial integration region$[\psi_0,\psi_0]\times[ð_0,ð_1]$, use Taylor's series for $\beta(\psi,ð)$ about$(\psi_1,ð_1)$,

$$\beta(\psi,ð) = \Big[1 + (\psi-\psi_1)D_\psi + (ð-ð_1)D_ð + \frac{(\psi-\psi_1)^2}{2!}D_\psi^2 + (\psi-\psi_1)(ð-ð_1)D_\psi D_ð$$
$$+ \frac{(ð-ð_1)^2}{2!}D_ð^2 + \frac{(\psi-\psi_1)^3}{3!}D_\psi^{(3)} + \frac{(\psi-\psi_1)^2(ð-ð_1)}{2!}D_\psi^2 D_ð$$
$$+ \frac{(\psi-\psi_1)(ð-ð_1)^2}{2!}D_\psi D_ð^2 + \frac{(ð-ð_1)^3}{3!}D_ð^3 + \frac{(\psi-\psi_1)^4}{4!}D_\psi^{(4)}$$
$$+ \frac{(\psi-\psi_1)^3(ð-ð_1)}{3!}D_\psi^{(3)}D_ð + \frac{(\psi-\psi_1)^2(ð-ð_1)^2}{2!\times 2!}D_\psi^{(2)}D_ð^{(2)}$$
$$+ \frac{(\psi-\psi_1)(ð-ð_1)^3}{3!}D_\psi D_ð^{(3)} + \frac{(ð-ð_1)^4}{4!}D_ð^{(4)} + \frac{(\psi-\psi_1)^5}{5!}D_\psi^{(5)}$$
$$+ \frac{(\psi-\psi_1)^4(ð-ð_1)}{4!}D_\psi^4 D_ð + \frac{(\psi-\psi_1)^3(ð-ð_1)^2}{3!\times 2!}D_\psi^3 D_ð^2$$
$$+ \frac{(\psi-\psi_1)^2(ð-ð_1)^3}{2!\times 3!}D_\psi^2 D_ð^3 + \frac{(\psi-\psi_1)(ð-ð_1)^4}{4!}D_\psi D_ð^4 + \frac{(ð-ð_1)^5}{5!}D_ð^{(5)}$$
$$+ \ldots\Big]\beta(\psi_1,ð_1) \qquad (2)$$

By integrate equation (2) on$(\psi_0,\psi_1)\times(ð_0,ð_1)$ we get:

$$\int_{ð_0}^{ð_1}\int_{\psi_0}^{\psi_1}\beta(\psi,ð)d\psi dð$$
$$= \Big[\gamma^2 - \frac{\gamma^3}{2}D_\psi - \frac{\gamma^3}{2}D_ð + \frac{\gamma^4}{6}D_\psi^2 + \frac{\gamma^4}{4}D_\psi D_ð + \frac{\gamma^4}{6}D_ð^2 - \frac{\gamma^5}{24}D_\psi^3 - \frac{\gamma^5}{12}D_\psi^2 D_ð - \frac{\gamma^5}{12}D_\psi D_ð^2$$
$$- \frac{\gamma^5}{24}D_ð^3 + \frac{\gamma^6}{120}D_\psi^4 + \frac{\gamma^6}{48}D_\psi^3 D_ð + \frac{\gamma^6}{36}D_\psi^2 D_ð^2 + \frac{\gamma^6}{48}D_\psi D_ð^3 + \frac{\gamma^6}{120}D_ð^4 - \frac{\gamma^7}{720}D_\psi^5$$
$$- \frac{\gamma^7}{240}D_\psi^4 D_ð - \frac{\gamma^7}{144}D_\psi^3 D_ð^2 - \frac{\gamma^7}{144}D_\psi^2 D_ð^3 - \frac{\gamma^7}{240}D_\psi D_ð^4 - \frac{\gamma^7}{720}D_ð^5$$
$$+ \cdots.\Big]\beta(\psi_1,ð_1) \qquad (3)$$

Substituting the points:

$$(\psi_o, ð_0), (\psi_0, ð_2), (\psi_0, ð_0+\gamma), \left(\psi_0, ð_0+\frac{1}{2}\gamma\right), \left(\psi_0, ð_0+\frac{3}{2}\gamma\right), (\psi_2, ð_0), (\psi_2, ð_2), (\psi_0, ð_2 + \gamma),$$
$$\left(\psi_0, ð_2+\frac{1}{2}\gamma\right), \left(\psi_0, ð_2+\frac{3}{2}\gamma\right), (\psi_0+\gamma, ð_0), (\psi_0+\gamma, ð_2), (\psi_0+\gamma, ð_0 + \gamma), \left(\psi_0+\gamma, ð_0+\frac{1}{2}\gamma\right), \left(\psi_0+\gamma, ð_0+\frac{3}{2}\gamma\right), \left(\psi_0+\frac{1}{2}\gamma, ð_0\right), \left(\psi_0+\frac{3}{2}\gamma, ð_0\right), \left(\psi_0 + \frac{1}{2}\gamma, ð_2\right), \left(\psi_0+\frac{3}{2}\gamma, ð_2\right), \left(\psi_0+\gamma, ð_0+\frac{1}{2}\gamma\right), \left(\psi_0+3h, ð_0+\frac{1}{2}\gamma\right), \left(\psi_0+\frac{1}{2}\gamma, ð_0 + \frac{1}{2}\gamma\right), \left(\psi_0+\frac{1}{2}\gamma, ð_0+\frac{3}{2}\gamma\right), \left(\psi_0+\frac{3}{2}\gamma, ð_0+\frac{1}{2}\gamma\right), \left(\psi_0+\frac{3}{2}\gamma, ð_0+\frac{3}{2}\gamma\right)$$

in equation (2) and adding the result to equation (3) to obtain:

$$\int_{ð_0}^{ð_1}\int_{\psi_0}^{\psi_1} \beta(\psi, ð)\,d\psi\, dð =$$
$$= \frac{121}{36}\gamma^2 \Big[4\beta(\psi_0, ð_0) + 4\beta(\psi_0, ð_1) + 8\beta(\psi_0, ð_0+\gamma) + 12\beta(\psi_0, ð_0+\gamma)$$
$$+ 2\beta\left(\psi_0, ð_0+\frac{1}{2}\gamma\right) + 2\beta\left(\psi_0, ð_0+\frac{3}{2}\gamma\right) + 4\beta(\psi_1, ð_0) + 4\beta(\psi_1, ð_1)$$
$$+ 12\beta(\psi_0, ð_1+\gamma) + 2\beta\left(\psi_0, ð_1+\frac{1}{2}\gamma\right) + 2\beta\left(\psi_0, ð_1+\frac{3}{2}\gamma\right) + 8\beta(\psi_0+\gamma, ð_1)$$
$$+ 12\beta(\psi_0+\gamma, ð_0) + 12\beta(\psi_0+\gamma, ð_1) + 36\beta(\psi_0+\gamma, ð_0+\gamma)$$
$$+ 6\beta\left(\psi_0+\gamma, ð_0+\frac{1}{2}\gamma\right) + 6\beta\left(\psi_0+\gamma, ð_0+\frac{3}{2}\gamma\right) + 2\beta\left(\psi_0+\frac{1}{2}\gamma, ð_0\right)$$
$$+ 2\beta\left(\psi_0+\frac{3}{2}\gamma, ð_0\right) + 2\beta\left(\psi_0+\frac{1}{2}\gamma, ð_1\right) + 2\beta\left(\psi_0+\frac{3}{2}\gamma, ð_1\right)$$
$$+ 6\beta\left(\psi_0+\gamma, ð_0+\frac{1}{2}\gamma\right) + 6\beta\left(\psi_0+3\gamma, ð_0+\frac{1}{2}\gamma\right) + \beta\left(\psi_0+\frac{1}{2}\gamma, ð_0+\frac{1}{2}\gamma\right)$$
$$+ \beta\left(\psi_0+\frac{1}{2}\gamma, ð_0+\frac{3}{2}\gamma\right) + \beta\left(\psi_0+\frac{3}{2}\gamma, ð_0+\frac{1}{2}\gamma\right) + \beta\left(\psi_0+\frac{3}{2}\gamma, ð_0+\frac{3}{2}\gamma\right)\Big]$$
$$+ \Big[-R_1\gamma^6\left(D_\psi^{(4)} + D_ð^{(4)}\right)$$
$$- R_2\gamma^8(R_3\left(D_\psi^{(6)}R_3D_ð^{(6)} - R_4D_\psi^{(5)}D_ð - R_4D_\psi D_ð^{(5)} - R_5D_\psi^{(4)}D_ð^{(2)} - R_5D_\psi^{(2)}D_ð^{(4)}\right)$$
$$+ R_2\gamma^9\left(-R_3D_\psi^{(7)} - R_3D_ð^{(7)} + \cdots\right)$$
$$+ \cdots\Big]\,\beta(\psi_1, ð_1) \qquad (4)$$

For the other three integrals in equation (1), the derivative of the function is continuous, so we can calculate their values and add them to equation (4) to get:

$$SySy = \int_{ð_0}^{ð_n}\int_{\psi_0}^{\psi_n}\beta(\psi,ð)d\psi dð =$$

$$= \frac{1}{36}\gamma^2\Bigg[25\big(\beta(a,c)+\beta(a,d)+\beta(b,c)+\beta(b,d)\big)+968\sum_{j=1}^{\frac{n}{2}-1}(\beta(a,c+j\gamma)+\beta(b,c+j\gamma)$$

$$+1452\sum_{j=1}^{\frac{n}{2}}\big(\beta(a,c+(2j-1)\gamma)+\beta(b,c+(2j-1)\gamma)\big)$$

$$+242\sum_{j=1}^{n}\left(\beta\left(a,c+\frac{2j-1}{2}\gamma\right)+\beta\left(b,c+\frac{2j-1}{2}\gamma\right)\right)$$

$$+121\sum_{i=1}^{\frac{n}{2}-1}\Bigg(8\beta(a+i\gamma,c)+8\beta(a+i\gamma,d)+16\sum_{j=1}^{\frac{n}{2}-1}\beta(a+ih,c+j\gamma)$$

$$+24\sum_{j=1}^{\frac{n}{2}}\beta(a+ih,c+(2j-1)\gamma)+4\sum_{j=1}^{n}\beta\left(a+ih,c+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+121\sum_{i=1}^{\frac{n}{2}}\Bigg(12\beta(a+(2i-1)\gamma,c)+12\beta(a+(2i-1)\gamma,b)+24\sum_{j=1}^{\frac{n}{2}-1}\beta(a+i\gamma,c+(2j-1)\gamma)$$

$$+36\sum_{j=1}^{\frac{n}{2}}\beta(a+(2i-1)\gamma,c+(2j-1)\gamma)+6\sum_{j=1}^{n}\beta\left(a+(2i-1)\gamma,c+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+121\sum_{i=1}^{n}\Bigg(2\beta\left(a+\frac{2i-1}{2}\gamma,c\right)+2\beta\left(a+\frac{2i-1}{2}\gamma,b\right)+4\sum_{j=1}^{\frac{n}{2}-1}\beta\left(a+i\gamma,c+\frac{2j-1}{2}\gamma\right)$$

$$+6\sum_{j=1}^{\frac{n}{2}}\beta\left(a+(2i-1)h,c+\frac{2j-1}{2}\gamma\right)+\sum_{j=1}^{n}\beta\left(a+\frac{2i-1}{2}\gamma,c+\frac{2j-1}{2}\gamma\right)\Bigg)\Bigg] \quad \ldots$$

$$+\left[-R_1\gamma^6\left(D_\psi^{(4)}+D_ð^{(4)}\right)-R_2\gamma^8(R_3\left(D_\psi^{(6)}R_3D_ð^{(6)}-R_4D_\psi^{(5)}D_ð-R_4D_\psi D_ð^{(5)}-R_5D_\psi^{(4)}D_ð^{(2)}-R_5D_\psi^{(2)}D_ð^{(4)}\right)\right.$$
$$\left.+R_2\gamma^9\left(-R_3D_\psi^{(7)}-R_3D_ð^{(7)}+\cdots\right)+\cdots\right]\beta(\psi_1,ð_1)+\lambda_{sy}\gamma^2+\epsilon_{sy}\gamma^4+\theta_{sy}\gamma^6+\cdots$$

such that $A_i, B_i, \ldots,\ i = 1,2,3$ are constant depend on the partial derivatives for the function $\beta(\psi,ð)$.

4. Finding the error formula using the suggested method $SySy$ to calculate the double integral with continuous integrands and improper derivative from the upper end

Theorem (2): Let the function $\beta(\psi,ð)$ be continuous and differentiable at each point of the integration region except at the point $(\psi,ð) = (\psi_n.ð_n)$ the approximate value of the double integral

$$J = \int_{ð_o}^{ð_n}\int_{\psi_0}^{\psi_n}\beta(\psi,ð)dxdy$$

can be evaluated using the following $SySy$ method

And the error formula is :

$$SySy = \int_{\$_0}^{\$_n}\int_{\psi_0}^{\psi_n}\beta(\psi,\$)d\psi d\$$$

$$= \frac{1}{36}\gamma^2\Bigg[25\big(\beta(a,c)+\beta(a,d)+\beta(b,c)+\beta(b,d)\big)+968\sum_{j=1}^{\frac{n}{2}-1}(\beta(a,c+j\gamma)+\beta(b,c+j\gamma)$$

$$+1452\sum_{j=1}^{\frac{n}{2}}\big(\beta(a,c+(2j-1)\gamma)+\beta(b,c+(2j-1)\gamma)\big)$$

$$+242\sum_{j=1}^{n}\left(\beta\left(a,c+\frac{2j-1}{2}\gamma\right)+\beta\left(b,c+\frac{2j-1}{2}\gamma\right)\right)$$

$$+121\sum_{i=1}^{\frac{n}{2}-1}\Bigg(8\beta(a+i\gamma,c)+8\beta(a+i\gamma,d)+16\sum_{j=1}^{\frac{n}{2}-1}\beta(a+ih,c+j\gamma)$$

$$+24\sum_{j=1}^{\frac{n}{2}}\beta(a+ih,c+(2j-1)\gamma)+4\sum_{j=1}^{n}\beta\left(a+i\gamma,c+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+121\sum_{i=1}^{\frac{n}{2}}\Bigg(12\beta(a+(2i-1)\gamma,c)+12\beta(a+(2i-1)\gamma,b)$$

$$+24\sum_{j=1}^{\frac{n}{2}-1}\beta(a+i\gamma,c+(2j-1)\gamma)+36\sum_{j=1}^{\frac{n}{2}}\beta(a+(2i-1)\gamma,c+(2j-1)\gamma)$$

$$+6\sum_{j=1}^{n}\beta\left(a+(2i-1)\gamma,c+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+121\sum_{i=1}^{n}\Bigg(2\beta\left(a+\frac{2i-1}{2}\gamma,c\right)+2\beta\left(a+\frac{2i-1}{2}\gamma,b\right)+4\sum_{j=1}^{\frac{n}{2}-1}\beta\left(a+i\gamma,c+\frac{2j-1}{2}\gamma\right)$$

$$+6\sum_{j=1}^{\frac{n}{2}}\beta\left(a+(2i-1)\gamma,c+\frac{2j-1}{2}\gamma\right)+\sum_{j=1}^{n}\beta\left(a+\frac{2i-1}{2}h,c+\frac{2j-1}{2}\gamma\right)\Bigg)\Bigg]$$

$$+\Big[-R_1\gamma^6\left(D_\psi^{(4)}+D_ð^{(4)}\right)-R_2\gamma^8(R_3\left(D_\psi^{(6)}R_3D_ð^{(6)}-R_4D_\psi^{(5)}D_ð-R_4D_\psi D_ð^{(5)}-R_5D_\psi^{(4)}D_ð^{(2)}-R_5D_\psi^{(2)}D_ð^{(4)}\right)$$

$$+R_2\gamma^9\left(-R_3D_\psi^{(7)}-R_3D_ð^{(7)}+\cdots\right)+\cdots\Big]\in'(\psi_1,ð_1)+\lambda'_{sy}\gamma^2+\in'_{sy}\gamma^4+\theta'_{sy}\gamma^6+\cdots$$

Where λ'_{sy} , $\in'_{sy}$, θ'_{sy} are constants that depend on the partial derivatives of the function.

Proof***:*** Suppose that the function$f(\psi, ð)$ is defined at each point of the integration region $[\psi_0, \psi_n] \times [ð_0, ð_n]$ and it is not improper and the partial derivatives of the function are not defined at the point $(\psi_n, ð_n)$.

Write the double integral J by:

$$J = \int_{ð_0}^{ð_n} \int_{\psi_0}^{\psi_n} \beta(\psi, ð) d\psi dð$$

$$= \int_{ð_0}^{ð_{n-1}} \int_{\psi_0}^{\psi_{n-1}} \beta(\psi, ð) d\psi dð + \sum_{s=0}^{n-2} \int_{ð_s}^{ð_{s+1}} \int_{\psi_{n-1}}^{\psi_n} \beta(\psi, ð) d\psi dð$$

$$+ \int_{ð_{n-1}}^{ð_n} \sum_{r=0}^{n-2} \int_{\psi_r}^{\psi_{r+1}} \beta(\psi, ð) d\psi dð + \int_{ð_{n-1}}^{ð_n} \int_{\psi_{n-1}}^{\psi_n} \beta(\psi, ð) d\psi dð \qquad \ldots (6)$$

Using the same steps in proof Theorem 1, we get:

$$SySy = \int_{ð_0}^{ð_n}\int_{\psi_0}^{\psi_n}\beta(\psi,ð)d\psi dð$$

$$= \frac{1}{36}\gamma^2\Bigg[25\big(\beta(a,c)+\beta(a,d)+\beta(b,c)+\beta(b,d)\big)+968\sum_{j=1}^{\frac{n}{2}-1}(\beta(a,c+j\gamma)+\beta(b,c+j\gamma)$$

$$+1452\sum_{j=1}^{\frac{n}{2}}\big(\beta(a,c+(2j-1)\gamma)+\beta(b,c+(2j-1)\gamma)\big)$$

$$+242\sum_{j=1}^{n}\left(\beta\left(a,c+\frac{2j-1}{2}\gamma\right)+\beta\left(b,c+\frac{2j-1}{2}\gamma\right)\right)$$

$$+121\sum_{i=1}^{\frac{n}{2}-1}\Bigg(8\beta(a+i\gamma,c)+8\beta(a+i\gamma,d)+16\sum_{j=1}^{\frac{n}{2}-1}\beta(a+i\gamma,c+j\gamma)$$

$$+24\sum_{j=1}^{\frac{n}{2}}\beta(a+i\gamma,c+(2j-1)\gamma)+4\sum_{j=1}^{n}\beta\left(a+i\gamma,c+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+121\sum_{i=1}^{\frac{n}{2}}\Bigg(12\beta(a+(2i-1)\gamma,c)+12\beta(a+(2i-1)\gamma,b)+24\sum_{j=1}^{\frac{n}{2}-1}\beta(a+ih,c+(2j-1)\gamma)$$

$$+36\sum_{j=1}^{\frac{n}{2}}\beta(a+(2i-1)\gamma,c+(2j-1)\gamma)+6\sum_{j=1}^{n}\beta\left(a+(2i-1)h,c+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+121\sum_{i=1}^{n}\Bigg(2\beta\left(a+\frac{2i-1}{2}\gamma,c\right)+2\beta\left(a+\frac{2i-1}{2}\gamma,b\right)+4\sum_{j=1}^{\frac{n}{2}-1}\beta\left(a+ih,c+\frac{2j-1}{2}\gamma\right)$$

$$+6\sum_{j=1}^{\frac{n}{2}}\beta\left(a+(2i-1)\gamma,c+\frac{2j-1}{2}\gamma\right)+\sum_{j=1}^{n}\beta\left(a+\frac{2i-1}{2}\gamma,c+\frac{2j-1}{2}\gamma\right)\Bigg)\Bigg]$$

$$+\Big[-R_1\gamma^6\left(D_\psi^{(4)}+D_ð^{(4)}\right)$$
$$-R_2\gamma^8(R_3\left(D_\psi^{(6)}R_3D_ð^{(6)}-R_4D_\psi^{(5)}D_ð-R_4D_\psi D_ð^{(5)}-R_5D_\psi^{(4)}D_ð^{(2)}-R_5D_\psi^{(2)}D_ð^{(4)}\right)$$
$$+R_2\gamma^9\left(-R_3D_\psi^{(7)}-R_3D_ð^{(7)}+\cdots\right)+\cdots\Big]\beta(\psi_{n-1},ð_{n-1})+\lambda'_{sy}\gamma^2+\in'_{sy}\gamma^4+\theta'_{sy}\gamma^6$$
$$+\cdots \qquad (7)$$

and the proof is complete.

5. Finding the error formula using the suggested method $SySy$ to calculate the double integral with continuous integrands and improper derivative from the upper and lower ends:

Theorem (3): Let the function $\beta(\psi,ð)$be continuous and differentiable at each point of the integration region except at the point $(\psi_n,ð_n)$, $(\psi_o,ð_0)$, the approximate value of the double integral

$$J = \int_{ð_0}^{ð_n} \int_{\psi_0}^{\psi_n} \beta(\psi, ð) d\psi dð$$

can be evaluated using the following $SySy$ method

$$
\begin{aligned}
SySy = \int_{\$_0}^{\$_n} \int_{\psi_0}^{\psi_n} \beta(\psi, \$) d\psi d\$ \\
= \frac{121}{36}\gamma^2 \Bigg[& 4\big(\beta(\psi_0, \$_0) + \beta(\psi_0, \$_n) + \beta(\psi_n, \$_o) + \beta(\psi_n, \$_n)\big) \\
& + 8\sum_{j=1}^{\frac{n}{2}-1} (\beta(\psi_0, \$_0 + j\gamma) + \beta(\psi_n, \$_0 + j\gamma) \\
& + 12\sum_{j=1}^{\frac{n}{2}} \big(\beta(\psi_0, \$_0 + (2j-1)\gamma) + \beta(\psi_0, \$_n + (2j-1)\gamma)\big) \\
& + 2\sum_{j=1}^{n} \left(\beta\left(\psi_0, \$_0 + \frac{2j-1}{2}\gamma\right) + \beta\left(\psi_0, \$_n + \frac{2j-1}{2}\gamma\right)\right) \\
& + \sum_{i=1}^{\frac{n}{2}-1} \Bigg(8\beta(\psi_0 + i\gamma, \$_0) + 8\beta(\psi_0 + i\gamma, \$_n) + 16\sum_{j=1}^{\frac{n}{2}-1} \beta(\psi_0 + i\gamma, \$_o + j\gamma) \\
& + 24\sum_{j=1}^{\frac{n}{2}} \beta(\psi_0 + i\gamma, \$_0 + (2j-1)\gamma) + 4\sum_{j=1}^{n} \beta\left(\psi_0 + i\gamma, \$_0 + \frac{2j-1}{2}\gamma\right)\Bigg) \\
& + \sum_{i=1}^{\frac{n}{2}} \Bigg(12\beta(\psi_0 + (2i-1)\gamma, \$_0) + 12\beta(\psi_0 + (2i-1)\gamma, \$_n) \\
& + 24\sum_{j=1}^{\frac{n}{2}-1} \beta(\psi_0 + i\gamma, \$_0 + (2j-1)\gamma) + 36\sum_{j=1}^{\frac{n}{2}} \beta(\psi_0 + (2i-1)\gamma, \$_0 + (2j-1)\gamma) \\
& + 6\sum_{j=1}^{n} \beta\left(\psi_0 + (2i-1)\gamma, \$_0 + \frac{2j-1}{2}\gamma\right)\Bigg) \\
& + \sum_{i=1}^{n} \Bigg(2\beta\left(\psi_0 + \frac{2i-1}{2}\gamma, \$_0\right) + 2\beta\left(\psi_0 + \frac{2i-1}{2}\gamma, \$_n\right) \\
& + 4\sum_{j=1}^{\frac{n}{2}-1} \beta\left(\psi_0 + i\gamma, \$_0 + \frac{2j-1}{2}\gamma\right) + 6\sum_{j=1}^{\frac{n}{2}} \beta\left(\psi_0 + (2i-1)\gamma, \$_0 + \frac{2j-1}{2}\gamma\right) \\
& + \sum_{j=1}^{n} \beta\left(\psi_0 + \frac{2i-1}{2}\gamma, \$_0 + \frac{2j-1}{2}\gamma\right)\Bigg)\Bigg]
\end{aligned}
$$

And the error formula is :

$$\Big[-R_1\gamma^6\Big(D_\psi^{(4)}+D_ð^{(4)}\Big)-R_2\gamma^8(R_3\Big(D_\psi^{(6)}R_3D_ð^{(6)}-R_4D_\psi^{(5)}D_ð-R_4D_\psi D_ð^{(5)}-R_5D_\psi^{(4)}D_ð^{(2)}-R_5D_\psi^{(2)}D_ð^{(4)}\Big)$$

$$+R_2\gamma^9\Big(-R_3D_\psi^{(7)}-R_3D_ð^{(7)}+\cdots\Big)+\cdots\Big]\ \beta(\psi_1,ð_1)$$

$$+\ \Big[-R_1\gamma^6\Big(D_\psi^{(4)}+D_ð^{(4)}\Big)$$

$$-R_2\gamma^8(R_3\Big(D_\psi^{(6)}R_3D_ð^{(6)}-R_4D_\psi^{(5)}D_ð-R_4D_\psi D_ð^{(5)}-R_5D_\psi^{(4)}D_ð^{(2)}-R_5D_\psi^{(2)}D_ð^{(4)}\Big)$$

$$+R_2\gamma^9\Big(-R_3D_\psi^{(7)}-R_3D_ð^{(7)}+\cdots\Big)+\cdots\Big]\ \beta(\psi_{n-1},ð_{n-1})+\lambda''_{sy}\gamma^2+\in''_{sy}\gamma^4$$

$$+\theta''_{sy}\gamma^6$$

Where $\lambda''_{sy}, \in''_{sy}, \theta''_{sy}$ .... are constants that depend on the partial derivatives of the function.

Proof***:*** We can rewrite the above integral as :

$$J=\int_{ð_0}^{ð_n}\int_{\psi_0}^{\psi_n}\beta(\psi,ð)d\psi dð$$

$$=\int_{ð_0}^{ð_1}\int_{\psi_0}^{\psi_1}\beta(\psi,ð)d\psi d\ +\sum_{s=0}^{n-1}\int_{ð_s}^{ð_{s+1}}\int_{\psi_0}^{\psi_1}\beta(\psi,ð)d\psi dð$$

$$+\sum_{s=1}^{n-1}\int_{ð_s}^{\psi_{s+1}}\sum_{r=0}^{n-2}\int_{\psi_i}^{\psi_{i+1}}\beta(\psi,ð)d\psi dð\ +\sum_{s=0}^{n-2}\int_{ð_z}^{ð_{z+1}}\int_{\psi_{n-1}}^{\psi_n}\beta(\psi,ð)d\psi dð$$

$$+\int_{ð_{n-1}}^{ð_n}\int_{\psi_{n-1}}^{\psi_n}\beta(\psi,ð)d\psi dð \qquad \ldots(8)$$

Similar to the steps in proofs of Theorems 1 and 2, we can evaluate the first and the last integrals in equation (8) respectively. For the other three integrals, the integrand is continuous derivatives on their integration region, so it is not difficult to calculate them and then add the result to equations (7) and (8) to obtain:

$$SySy = \int_{ð_0}^{ð_n}\int_{\psi_0}^{\psi_n}\beta(\psi,ð)d\psi dð$$

$$= \frac{121}{36}\gamma^2\Bigg[\ 4\big(\beta(\psi_0,ð_0)+\beta(\psi_0,ð_n)+\beta(\psi_n,ð_o)+\beta(\psi_n,ð_n)\big)$$

$$+8\sum_{j=1}^{\frac{n}{2}-1}(\beta(\psi_0,ð_0+j\gamma)+\beta(\psi_n,ð_0+j\gamma)$$

$$+12\sum_{j=1}^{\frac{n}{2}}\big(\beta(\psi_0,ð_0+(2j-1)\gamma)+\beta(\psi_0,ð_n+(2j-1)\gamma)\big)$$

$$+2\sum_{j=1}^{n}\left(\beta\left(\psi_0,ð_0+\frac{2j-1}{2}\gamma\right)+\beta\left(\psi_0,ð_n+\frac{2j-1}{2}\gamma\right)\right)$$

$$+\sum_{i=1}^{\frac{n}{2}-1}\Bigg(\ 8\beta(\psi_0+i\gamma,ð_0)+8\beta(\psi_0+i\gamma,ð_n)+16\sum_{j=1}^{\frac{n}{2}-1}\beta(\psi_0+i\gamma,ð_o+j\gamma)$$

$$+24\sum_{j=1}^{\frac{n}{2}}\beta(\psi_0+i\gamma,ð_0+(2j-1)\gamma)+4\sum_{j=1}^{n}\beta\left(\psi_0+i\gamma,ð_0+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+\sum_{i=1}^{\frac{n}{2}}\Bigg(12\beta(\psi_0+(2i-1)\gamma,ð_0)+12\beta(\psi_0+(2i-1)\gamma,ð_n)$$

$$+24\sum_{j=1}^{\frac{n}{2}-1}\beta(\psi_0+i\gamma,ð_0+(2j-1)\gamma)+36\sum_{j=1}^{\frac{n}{2}}\beta(\psi_0+(2i-1)\gamma,ð_0+(2j-1)\gamma)$$

$$+6\sum_{j=1}^{n}\beta\left(\psi_0+(2i-1)\gamma,ð_0+\frac{2j-1}{2}\gamma\right)\Bigg)$$

$$+\sum_{i=1}^{n}\Bigg(\ 2\beta\left(\psi_0+\frac{2i-1}{2}\gamma,ð_0\right)+\ 2\beta\left(\psi_0+\frac{2i-1}{2}h,ð_n\right)$$

$$+\ 4\sum_{j=1}^{\frac{n}{2}-1}\beta\left(\psi_0+ih,ð_0+\frac{2j-1}{2}\gamma\right)+6\sum_{j=1}^{\frac{n}{2}}\beta\left(\psi_0+(2i-1)\gamma,ð_0+\frac{2j-1}{2}\gamma\right)$$

$$+\sum_{j=1}^{n}\beta\left(\psi_0+\frac{2i-1}{2}\gamma,ð_0+\frac{2j-1}{2}\gamma\right)\Bigg)\Bigg]$$

$$+\Big[-R_1\gamma^6\left(D_\psi^{(4)}+D_ð^{(4)}\right)$$

$$-R_2\gamma^8(R_3\left(D_\psi^{(6)}R_3D_ð^{(6)}-R_4D_\psi^{(5)}D_ð-R_4D_\psi D_ð^{(5)}-R_5D_\psi^{(4)}D_ð^{(2)}-R_5D_\psi^{(2)}D_ð^{(4)}\right)$$

$$+R_2\gamma^9\left(-R_3D_\psi^{(7)}-R_3D_ð^{(7)}+\cdots\right)+\cdots\Big]\ \beta(\psi_1,ð_1)$$

$$+ \ \Big[-R_1\gamma^6\left(D_\psi^{(4)}+D_ð^{(4)}\right) - R_2\gamma^8(R_3\left(D_\psi^{(6)}R_3D_ð^{(6)} - R_4D_ð^{(5)}D_ð - R_4D_\psi D_ð^{(5)} - R_5D_\psi^{(4)}D_ð^{(2)} - R_5D_\psi^{(2)}D_ð^{(4)}\right) + R_2\gamma^9\left(-R_3D_\psi^{(7)} - R_3D_ð^{(7)} + \cdots\right) + \cdots\Big] \ \beta(\psi_{n-1}, ð_{n-1}) + \lambda''_{sy}\gamma^2 + \in''_{sy}\gamma^4 + \theta''_{sy}\gamma^6$$

Thus the theorem is established.

6. Romberg Accelerating

Werner Werner (1909–2003) introduced this method in 1955, where the trigonometric arrangement consists of numerical approximations of the definite integral by repeatedly applying the Richardson outer modification to the trapezoidal base, the midpoint rule, or the Simpson rule, which represent some of the Newton-Coates formulas. The general formula for this acceleration is:

$$\alpha = \frac{\left(2^L\alpha\left(\frac{k}{2}\right) - \alpha(k)\right)}{\left(2^L - 1\right)}$$

Such that $L = 4,6,8,10,\ldots$ (in Simpson's rule) and $L = 2,4,6,8,10,\ldots$ (in Trapezoidal rule and Mid-point rule), α is the value in that we written it in a new column in tables whereas $\alpha(k)$ and $\alpha\left(\frac{k}{2}\right)$ are values in its previous column. Moreover, the first column from the table represents one of Newton Coates values [3].

7. Example and Results

1-The function of $\int_0^1\int_0^1 ð\sqrt{\varphi + ð}\ dðd\varphi$ shown in Figure (7-1) is continuous in the region $[0,1]\times[0,1]$ and differentiable in the region $(0,1]\times(0,1]$, but the partial derivatives are not defined at the point$(0,0)$, i.e., the integer has an imperfect derivative at the lower limit, and the type of imperfection is radical. Therefore, the error formulas (correction limits) for the integer $I = \int_0^1\int_0^1 ð\sqrt{\varphi + ð}\ d\varphi dð$ using the rule $(SySy)$ and using the formulas in the theorems (3.2.1.1) respectively, are as follows:

$$E_{SySy}(\gamma) = A_{SySy}\gamma^2 + a_1\gamma^{3.5} + B_{SySy}\gamma^4 + C_{SySy}\gamma^6 + \cdots$$

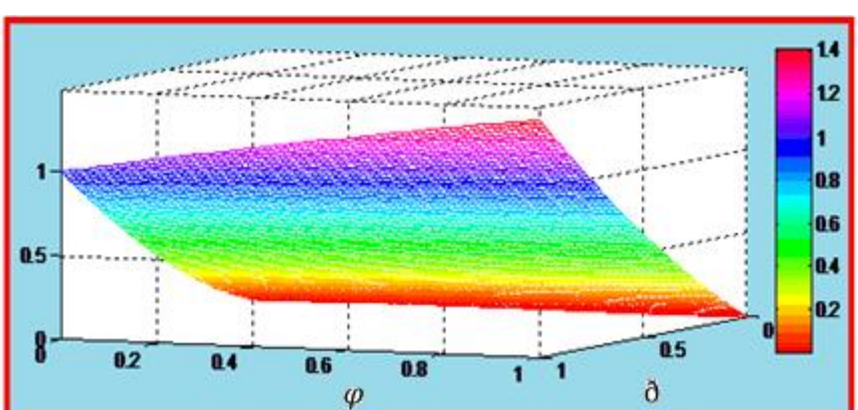


Figure (7 -1) shows the function diagram $\beta(\varphi, ð) = ð\sqrt{\varphi + ð}$

We conclude from Table (7 -1) that when $n = 64$ the value of the integration using the $(SySy)$ rule is correct to four decimal places, while the value using the $(RSySy)$ method is identical to the exact value (rounded to fourteen decimal places). Musa [24] obtained the same result and the same number of subintervals when using the (RSS) method.

Table (7-1) double Integral Calculation $I = \int_0^1 \int_0^1 ð\sqrt{\varphi + ð}\ d\varphi dð$ In a way$RSySy$

N	$RSySy$
2	0.53269156536377
4	0.53222081164265
8	0.53221194544226
16	0.53221191434215
32	0.53221191422794
64	0.53221191422770

2- The function $\beta(\varphi, ð) = 4\sqrt{1 - \varphi ð}$ shown in Figure (7-2) is continuous in the region $[0,1] \times [0,1]$ and differentiable in the region $[0,1) \times [0,1)$, but the partial derivatives are not defined at the point $(1,1)$, i.e., the integer has an imperfect derivative at the upper limit and the type of imperfection is radical. Therefore, the error formulas (correction limits) for the integer $I = \int_0^1 \int_0^1 ð\sqrt{1 - \varphi ð}\ d\varphi dð$ using the rules $(SySy\ )$and are using the theorems in the results (3.2.3.1), (3.2.3.2) respectively, are as follows:

$$E_{SySy}(\gamma) = A_{SySy}\gamma^2 + a_1\gamma^{2.5} + a_2\gamma^{3.5} + B_{SySy}\gamma^4 + a_3\gamma^{4.5} + a_4\gamma^{5.5} + C_{SySy}\gamma^6 + \cdots$$

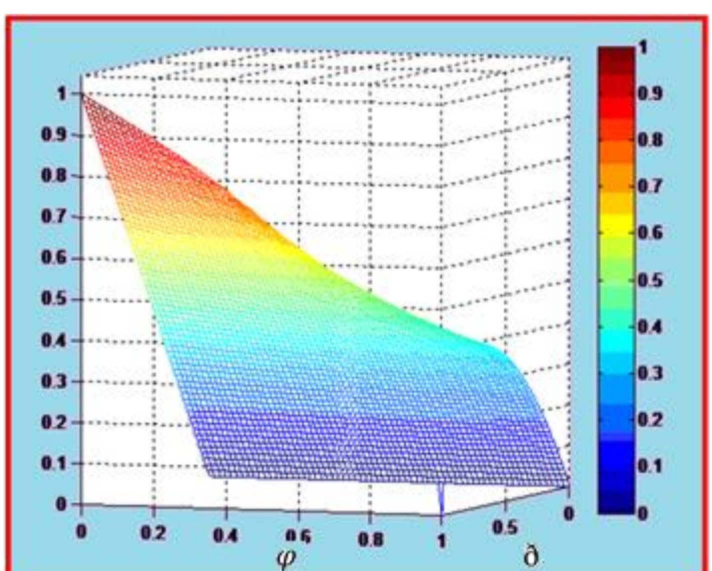


Figure (7-2) shows the function diagram $\beta(\varphi, ð) = ð\sqrt{1 - ð\varphi}$

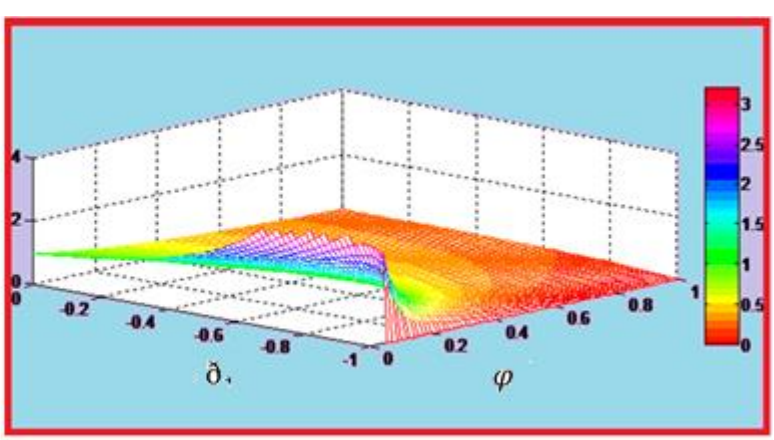
Table (7-2) double Integral Calculation $I = \int_0^1 \int_0^1 \sqrt{1-\varphi ð)}\ d\varphi dð$ In a way$RSySy$

N	$RSySy$
2	0.39363455326097
4	0.39996759521276
8	0.39999825084517
16	0.39999993845027
32	0.39999999815715
64	0.39999999996384
128	0.39999999999954
256	0.39999999999999

The results of this method are shown in Table (7-2) and the following has been shown: When $n = 256$, the value of the integration using the $(SySy)$ rule is correct to five decimal places, while the value using the $(RSySy)$ method is identical to the exact value.

3- The function $\beta(\varphi, ð) = \frac{1+ð}{(1+\varphi+ð)^2}$ shown in Figure (6-3) is undefined at the point $(0, -1)$ (lower limit of integration), i.e., it is defective, and the type of defect is relative. Therefore, the error formulas (correction limits) for the integration $I = \int_{-1}^0 \int_0^1 \frac{1+ð}{(1+\varphi+ð)^2}\ d\varphi dð$ using the rules $(SySy$ and using the theorems in the results (3.2.1.1) respectively, are as follows:

$E_{SySy}(\gamma) = a_1\gamma + A_{SySy}\gamma^2 + B_{SySy}\gamma^4 + C_{SySy}h^6 + \cdots$



The function is illustrated in Figure (7-3) $\beta(\varphi, ð) = \frac{1+ð}{(1+\varphi+ð)^2}$

Table (7-3) double Integral Calculation $I = \int_{-1}^{0}\int_{0}^{1}\frac{1+ð}{(1+\varphi+ð)^2}\,dxdy$ In a way $RSySy$

N	$RSySy$
2	0.55324074074074
4	0.69401455026455
8	0.69315884218662
16	0.69314728807120
32	0.69314718102450
64	0.69314718056073
128	0.69314718055995

We conclude from Table (7-3) that when $n = 128$ the value of the integration using the $(SySy)$ rule is correct for two decimal places, while the value using the $(RSySy)$ method is correct for thirteen decim

4- The function $\beta(\varphi, ð) = \sqrt{\varphi^4 + ð^4}$ shown in Figure (6-4) is continuous in the region $[0,1] \times [0,1]$ and differentiable in the region $(0,1] \times (0,1]$, but the partial derivatives are not defined at the point $(0,0)$, i.e., the integer has an imperfect derivative at the lower limit, and the type of imperfection is radical. Therefore, the error formulas (correction limits) for the integration using the rules $(SySy$ are, using the formulas in the theorem (3-1, as follows:

$$E_{SySy}(\gamma) = A_{SySy}\gamma^2 + (B_{SySy} + a_1)\gamma^4 + C_{SySy}\gamma^6 + \cdots$$

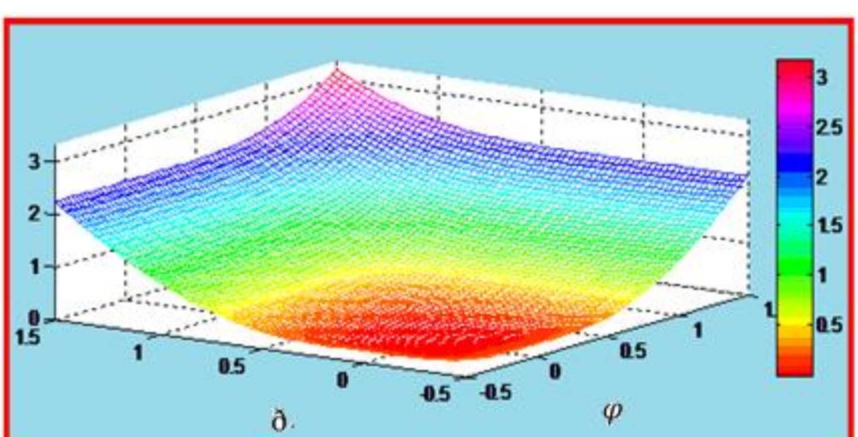


Figure (7-4) shows the function diagram $\beta(\varphi, ð) = \sqrt{\varphi^4 + ð^4}$

Table (7-4) double Integral Calculation $I = \int_0^1 \int_0^1 \sqrt{\varphi^4 + ð^4}\ d\varphi dð$ In a way$RSySy$

N	$RSvSv$
2	0.53659108508613
4	0.54475164413550
8	0.54471417522663
16	0.54471470994157
32	0.54471470660521
64	0.54471470661241
128	0.54471470661241

From Table (7-4), we observe that the value is constant for fourteen decimal places horizontally when $n = 128\ \ and\ \ k = 6, 8, 10, 12, 14$.

5- The function $\beta(\varphi, ð) = \sqrt{\varphi + ð}$ shown in Figure (7-5) is the integral, but the partial derivatives are abnormal at the point $(\varphi, ð) = (0,0)$.

$E_{SySy} = A_{SySy}\gamma^2 + a\gamma^{2.5} + B_{SySy}\gamma^4 + C_{SySy}\gamma^6 + D_{SySy}\gamma^8 + \cdots$

The type of malformation is radical. From applying theorems (3.1), we found that the correction limits are where $A_{SySy}, B_{SySy}, \ldots, a, b, c, and, d$ are constants.

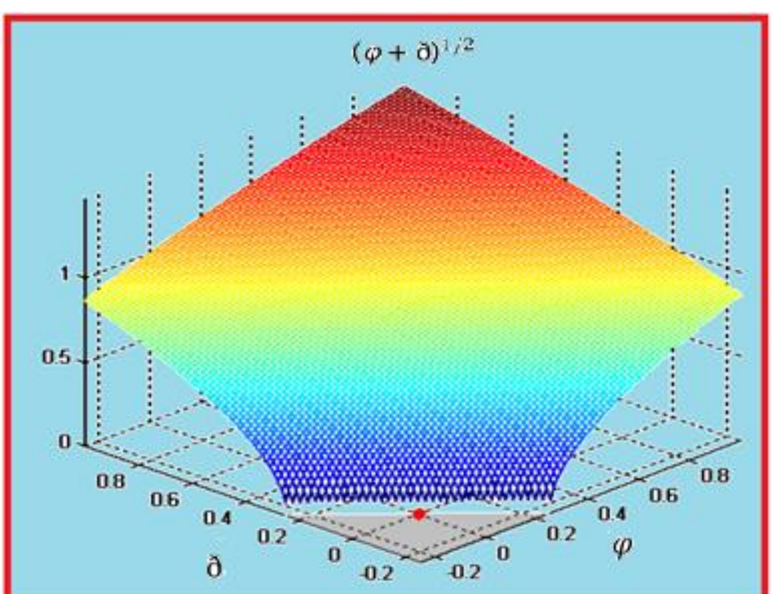


Figure (7-5) shows the function diagram $\beta(\varphi, ð) = \sqrt{\varphi + ð}$

Table (7-5) double Integral Calculation $I = \int_0^1 \int_0^1 \sqrt{\varphi + \eth}\, d\eth d\varphi$ In a way$RSySy$

	$RSvSv$
2	0.96858407730551
4	0.97512653518785
8	0.97516090179001
16	0.97516113191046
32	0.97516113319425
64	0.97516113319796

From Table (7-5), we conclude that when $n = 64$, the value of the integration using the rule ($SySy$) is correct to three decimal places, while the value using the method ($RSySy$) is correct to thirteen decimal places.

8. Conclusions

We conclude that the values of double integrals using SySy method give the correct values of several decimal digits compared with the exact values of the integrals using a number of partial periods without using a method of teasing. Moreover the tables show that the results will be better with a few relatively partial periods as well as the values are correct for several decimal digits which are between thirteen and fourteen correct decimal digits, when we use the Romberg's acceleration with the SySu method accompanying with the correction terms. In addition, the Romberg acceleration without ignore the impropriety will play an importance role to improve results in terms of accuracy and speed of approach to the real value of integrations with a few partial periods. Therefore we can use SySy method to evaluate the double integral whether the integrand is continuous or improper.

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