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Higher-Degree Fuzzy Polynomial Regression Model Estimation Stability: A Comparative Study

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ABSTRACT

Estimating fuzzy nonlinear regression models is an essential problem when observed responses are inaccurate and represented by fuzzy numbers. Fuzzy least squares regression based on shape conservation (FLSR-SPO) and the proposed ridge-based improved fuzzy nonlinear regression (PR-FNR) are two estimators that are available for this problem.

In this study, three sample sizes (n = 25, 75, 150) and two error variance levels ($\sigma^2 = 0.5, 2.5$) are simulated across two true polynomial scores (k = 3, 4). The estimators' performance was assessed using the root mean squared fuzzy error on independent, noise-free test sets after the experiment was performed 1000 times.

The results of the experiment indicate that, under all experimental settings, with increasing polynomial score and decreasing sample size, the PR-FNR obtains a much lower RMSE than the FLSR-SPO. These results show that the estimation instability caused on by higher-score fuzzy polynomial models is well reduced by the cross-validated ridge penalty implemented in the PR-FNR method.

1. Introduction

One of the most common problems in the applied sciences is the existence of imprecision in empirical data. The assumptions that underpin classical statistical modelling are inconsistent with observations, which are often rounded, recorded with instruments of low accuracy, or expressed through ambiguous linguistic descriptions. By modelling each inaccurate variable as a fuzzy number with a center value and left and right spreads that measure its possible deviation, fuzzy set theory [1] provides a logical mathematical framework for handling such uncertainty. When an explanatory variable is used to explain a fuzzy response, the ordinary regression model needs to be generalized to a fuzzy regression model with fuzzy quantities for the inputs and coefficients. Since Tanaka et al. [3] developed the possibilistic and Diamond [4] developed the fuzzy least squares approach, fuzzy linear regression gained a lot of study interest. On the other hand, the nonlinear framework presents two overlapping difficulties [5, 8, 13, 15]: the polynomial design matrix becomes more difficult as its degree increases, and fuzzy computation that transfers the diffusion via a nonlinear transformation typically fails to preserve the shape of the underlying fuzzy numbers. Two distinct approaches are employed to get over these difficulties. The first, Hong, Hwang, and Do's fuzzy least-squares regression based on shape-preserving operations (FLSR-SPO) [11] that uses the weakest triangular norm in order that the arithmetic on L–R fuzzy numbers again yields L–R numbers, then the Diamond method is used to minimize the variance to determine the center and spread. The second is the suggested improved ridge-based fuzzy nonlinear regression (PR-FNR) [14], which is part of a larger stream that integrates reduction and regularization into fuzzy regression [12, 16, 17], and [18]. It enhances the fuzzy least-squares with a ridge penalty whose magnitude is automatically determined by cross-validation and estimates the center along with the two spreads in a regularized, iteratively reweighting manner.

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In addition to these two estimators, recent studies have advanced least-squares fuzzy regression formulations [6, 7, 19, 20], and fuzzy regression approaches have proven useful in applied medical and epidemiological research [21, 22, 23]. In this study, a relevant topic is addressed: how do these two estimators perform when the underlying model takes a higher degree, a case where an unregularized least-squares fit is theoretically supposed to become unstable and the design matrix is significantly more unconditioned?. By identifying the two leading estimators and increasing the polynomial degree to three and four through a closely controlled experiment, the study answers this question.

2. The Two Estimation Methods

In this section, the fuzzy response is represented by $\tilde{Y}_i$, and a single explanatory variable X is taken into consideration. For j = 0, 1, …, k, the fuzzy regression coefficients are expressed as $\tilde{A}_j$ with lower, center, and upper components β_j^L, β_j^c and β_j^U.

2.1 FLSR-SPO Method

This method, which was initially put forth by Hong, Hwang, and Do [11], uses a shape- preserving arithmetic based on the weakest t-norm (T_ω-based) to carry out least-squares fuzzy polynomial regression for L–R fuzzy input–output data. Fuzzy arithmetic is simplified and a mixed nonlinear program is used to derive the solution since the weakest t-norm [7] Tω maintains the shape of fuzzy numbers under addition and multiplication. The t-norm that is weakest is:

$$T_\omega(a,b) = \begin{cases} a, \; if \; b \; = \; 1 \\ b, \; if \; a \; = \; 1 \\ 0, \; otherwise \end{cases} \tag{1}$$

Let F_{LRR} be the set of all L–R fuzzy numbers. The model under consideration is

$$P\colon \; \tilde{Y}_i \; = \; \sum_{j=1}^{p} \tilde{A}_j \; \otimes \; X^j \; \oplus \; \sum_{1 \le l \le k \le p} \tilde{A}_{l,k} \; \otimes \; X^l \; \otimes \; X^k \tag{2}$$

Where $\tilde{A}_j$, $\tilde{A}_{l,k}$, X, $\tilde{Y}_i$ ∈ F_{LRR}. The model reduces to the polynomial of degree k for a single independent variable X,

$$\tilde{Y}_i \; = \; \tilde{A}_0 \; \oplus \; \tilde{A}_1 \; \otimes \; X_i \; \oplus \; \tilde{A}_2 \; \otimes \; X_i^2 \; \oplus \; \cdots \; \oplus \; \tilde{A}_k \; \otimes \; X_i^k \tag{3}$$

Diamond's metric [4] on L–R fuzzy numbers is used to measure the fit. For $\tilde{A}_1 = (\beta_1^c, \beta_1^l, \beta_1^r)_{LR}$ and $\tilde{A}_2 = (\beta_2^c, \beta_2^l, \beta_2^r)_{LR}$,

$$D_{LR}^2(\tilde{A}_1, \tilde{A}_2) = (\beta_1^c - \beta_2^c)^2 + \left[\left(\beta_1^c - \beta_1^l\right) - \left(\beta_2^c - \beta_2^l\right)\right]^2 + [(\beta_1^c + \; \beta_1^r) - (\beta_2^c + \beta_2^r)]^2 \tag{4}$$

The least-squares problem, given the data pairs $(X_i, \tilde{Y}_i)$, i = 1, 2, …, n, is

$$D\colon min \, r(\beta) = \sum_{i=1}^{n} D_{LR}\left(\sum_j \tilde{A}_j \otimes X_{ij} \; \oplus \; \sum_{l,k} \tilde{A}_{l,k} \otimes X^l \otimes X_{ik} \, , \tilde{Y}_i\right) \tag{5}$$

The objective is quadratic on each area with M = {(j, l, k) | 1 ≤ j ≤ p, 1 ≤ l ≤ k ≤ p} and index functions f and g mapping to the cases H_1, ..., H_8 that determine which argument attains the spread maximum. The solution is

$$min \, r(\beta) \; = \; min_{f,g} \; min_{\beta \in \cap} \; r(\beta) \tag{6}$$

2.2 Proposed Ridge Fuzzy Nonlinear Regression (PR-FNR)

The proposed ridge fuzzy nonlinear regression (PR-FNR) method is a developed methodological extension of the ridge-regularized fuzzy nonlinear regression formulation of Farnoosh, Ghasemian, and Solaymani Fard [14], introduces structural and computational improvements that enhance estimation accuracy and stability through ridge regularization. Polynomial basis expansion, ridge regularization, adaptive iterative estimation with variance updates, non-negativity constraints on the spreads, and K-fold cross-validation for regularization parameter selection are all incorporated into the method. The unified model over L–R fuzzy numbers for each observation i = 1, 2, …, N is

$$\tilde{Y}_i = f(X_i;\, A) + \varepsilon_i \tag{7}$$

With $\tilde{Y}_i = (\tilde{Y}_i^c, \tilde{Y}_i^l, \tilde{Y}_i^r)_{LR}$, $X_i = (X_i^c, X_i^l, X_i^r)_{LR}$, fuzzy parameter $A = (\beta^c, \beta^l, \beta^r)$, and ε_i a fuzzy random error. The regression function is created by constructing a polynomial basis vector of degree k for each component $s \in \{c, l, r\}$:

$$\varphi(X_i^s) = [1, X_i^s, (X_i^s)^2, \dots, (X_i^s)^k], \quad f(X_i^s;\, \beta_s) = \beta_s\, \varphi(X_i^s) \tag{8}$$

The design matrix $\Phi_s \in \mathbb{R}^{N\times(k+1)}$ whose i-th row equals $\varphi(X_i^s)$, is assemble by the basis vectors. An adaptive iterative ridge approach is then used to jointly update the parameters and the residual variance. The parameters at iteration (t) are adjusted by initializing the error variance $h_s^{2(0)} > 0$:

$$\beta_s^{(t)}(\lambda|k) = \left[\Phi^T\Phi + N\lambda\, h_s^{2(t-1)}\, I\right]^{-1} \Phi^T \tilde{Y}_s \tag{9}$$

Next, the variance update:

$$h_s^{2(t)}(\lambda|k) = \frac{1}{N}\left(\tilde{Y}_s - \Phi\beta_s^{(t)}\right)^T \left(\tilde{Y}_s - \Phi\beta_s^{(t)}\right) \tag{10}$$

Iterating until $\|\beta_s^{(t)} - \beta_s^{(t-1)}\| < \varepsilon$. The fitted spreads' non-negativity is enforced by

$$\hat{Y}_i = \left(\beta^c\, \varphi(X_i^c), max(0, \beta^l\, \varphi(X_i^l)), max(0, \beta^r\, \varphi(X_i^r))\right)_{LR} \tag{11}$$

The fuzzy root mean square error (RMSE), which aggregates the squared deviations over the three fuzzy components, is used along with K-fold cross-validation to evaluate and adjust the regularization parameter:

$$RMSE(\lambda|k) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\left(\tilde{Y}_i^c - \hat{Y}_i^c\right)^2 + \left(\tilde{Y}_i^l - \hat{Y}_i^l\right)^2 + \left(\tilde{Y}_i^r - \hat{Y}_i^r\right)^2\right]} \tag{12}$$

$$CV(\lambda|k) = \frac{1}{K}\sum_{j=1}^{K} RMSE_{-j}(\lambda|k), \quad \lambda^* = argmin_\lambda\, CV(\lambda|k) \tag{13}$$

This data-driven penalty is exactly that which sets PR-FNR different from FLSR-SPO. In equation (9), the term $N\lambda h_s^2 I$ reduces the variance of the predicted spreads at the cost of a small bias by stabilizing the inversion of the near-singular Gram matrix $\Phi^T\Phi$ and decreasing the high-order coefficients toward zero.

3 Comparison Criterion

The fuzzy root mean square error (RMSE), which combines the squared differences between the observed center, upper, and lower values of the fuzzy response and their estimated equivalents, is used to compare the estimators of (FLSR-SPO) and (PR-FNR) on an independent test sample of size n.

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{Y}_i^c - \hat{Y}_i^c\right)^2 + \left(\tilde{Y}_i^U - \hat{Y}_i^U\right)^2 + \left(\tilde{Y}_i^L - \hat{Y}_i^L\right)^2} \quad (14)$$

Where $\tilde{Y}_i^c$, $\tilde{Y}_i^U$, $\tilde{Y}_i^L$ are the observed centre, upper and lower values and $\hat{Y}_i^c$, $\hat{Y}_i^U$, $\hat{Y}_i^L$ are the corresponding estimations. Since a smaller RMSE indicates a closer fit, the estimator that achieves the lower RMSE is considered more accurate.

4 Simulation Design

With the goal to determine the true parameters and precisely quantify the accuracy of each estimator, the data is generated using a fully specified fuzzy polynomial. The deterministic center is the polynomial $c(X) = \Sigma_j \beta_j X^j$ with true center coefficients β = (5, 3, 2, 1) for the degree-3 model and β = (5, 3, 2, 1, 0.5) for the degree-4 model. The explanatory variable X is sampled from a uniform distribution. Each training observation is given a triangular fuzzy number by adding a zero-mean Gaussian error of variance σ2 to its center and attaching symmetric left and right spreads equal to five percent of the absolute center. As a way for the test fuzzy numbers to function as precise goals in equation (14), the test sample is produced from the same model without noise. Table 1 provides an overview of the complete configuration for the experiment.

Table 1. Experiment factors and fixed parameters

Factor / setting	**Value(s)**
True model degree (k)	3 and 4
Error variance (σ^2)	0.5 and 2.5
Training sample size (n)	25, 75, 150
Replications (R)	1000
Test sample size	100 (noise-free)
True coefficients (k = 3)	5, 3, 2, 1
True coefficients (k = 4)	5, 3, 2, 1, 0.5
Comparison criterion	RMSE

5 Results and Analysis

This section introduces fuzzy directed graph theory and fuzzy topology basic definitions and preliminaries. All of fuzzy concepts are widely used and are found in references like [3, 5, 6, 7, 11].

5.1 Degree-3 Model

The averaged fuzzy-coefficient estimates and their standard deviations (given beneath each estimate in parenthesis) are recorded for the three sample sizes simultaneously for each combination of degree and error variance, with the averaged RMSE indicated in the terminal row. Next, the corresponding RMSE-versus-sample-size curves are displayed.

Table 2. RMSE and average fuzzy-coefficient estimations for the degree-3 model with σ² = 0.5

Parameter	FLSR-SPO			PR-FNR		
	n = 25	**n = 75**	**n = 150**	**n = 25**	**n = 75**	**n = 150**
$\boldsymbol{\beta_0^L}$	4.9172 (0.2504)	4.9383 (0.1412)	4.9674 (0.0936)	4.5026 (0.2702)	4.5529 (0.1419)	4.5566 (0.0981)
$\boldsymbol{\beta_0^c}$	4.9996 (0.2273)	4.9973 (0.1244)	4.9987 (0.0849)	4.9383 (0.2469)	4.9897 (0.1257)	4.9974 (0.0849)
$\boldsymbol{\beta_0^U}$	5.0820 (0.2635)	5.0563 (0.1384)	5.0301 (0.0977)	5.0097 (0.2628)	5.0444 (0.1341)	5.0482 (0.0909)
$\boldsymbol{\beta_1^L}$	2.9251 (0.1940)	2.9662 (0.0972)	2.9738 (0.0635)	3.1538 (0.3079)	3.1895 (0.1718)	3.1987 (0.1187)
$\boldsymbol{\beta_1^c}$	2.9971 (0.1666)	2.9980 (0.0925)	2.9989 (0.0629)	2.9743 (0.1679)	2.9955 (0.0928)	2.9980 (0.0630)
$\boldsymbol{\beta_1^U}$	3.0690 (0.2011)	3.0299 (0.0994)	3.0240 (0.0643)	3.2161 (0.1797)	3.2368 (0.0983)	3.2404 (0.0658)
$\boldsymbol{\beta_2^L}$	1.9094 (0.0851)	1.8583 (0.0704)	1.8228 (0.0368)	1.7360 (0.1487)	1.7221 (0.0814)	1.7179 (0.0536)
$\boldsymbol{\beta_2^c}$	1.9998 (0.0368)	2.0003 (0.0178)	1.9994 (0.0121)	2.0067 (0.0384)	2.0011 (0.0179)	1.9996 (0.0121)
$\boldsymbol{\beta_2^U}$	2.0902 (0.0823)	2.1424 (0.0680)	2.1761 (0.0371)	2.2075 (0.0414)	2.2074 (0.0190)	2.2070 (0.0128)
$\boldsymbol{\beta_3^L}$	0.9775 (0.0341)	0.9881 (0.0256)	0.9982 (0.0123)	0.9779 (0.0283)	0.9764 (0.0152)	0.9766 (0.0097)
$\boldsymbol{\beta_3^c}$	1.0007 (0.0175)	1.0002 (0.0090)	1.0002 (0.0060)	1.0028 (0.0176)	1.0004 (0.0090)	1.0003 (0.0061)
$\boldsymbol{\beta_3^U}$	1.0238 (0.0361)	1.0123 (0.0254)	1.0022 (0.0122)	1.0170 (0.0194)	1.0146 (0.0096)	1.0142 (0.0064)
RMSE	1.4778 (0.2061)	1.3380 (0.1018)	1.3134 (0.0878)	0.6963 (0.2323)	0.4863 (0.0955)	0.4345 (0.0673)

Table 3. RMSE and average fuzzy-coefficient estimations for the degree-3 model with σ² = 2.5

Parameter	FLSR-SPO			PR-FNR		
	n = 25	**n = 75**	**n = 150**	**n = 25**	**n = 75**	**n = 150**
$\boldsymbol{\beta_0^L}$	4.9090 (0.4985)	4.9333 (0.2876)	4.9619 (0.1945)	4.3662 (0.5601)	4.5196 (0.2938)	4.5393 (0.1980)
$\boldsymbol{\beta_0^c}$	4.9936 (0.4944)	4.9994 (0.2799)	4.9966 (0.1911)	4.8100 (0.5517)	4.9628 (0.2870)	4.9840 (0.1926)
$\boldsymbol{\beta_0^U}$	5.0782 (0.5186)	5.0655 (0.2906)	5.0314 (0.1997)	4.8844 (0.5720)	5.0221 (0.2984)	5.0393 (0.2008)
$\boldsymbol{\beta_1^L}$	2.9297 (0.3910)	2.9659 (0.2085)	2.9735 (0.1395)	3.1487 (0.4774)	3.1979 (0.2550)	3.2044 (0.1744)
$\boldsymbol{\beta_1^c}$	3.0025 (0.3829)	3.0033 (0.2053)	3.0000 (0.1405)	2.9375 (0.3904)	2.9913 (0.2061)	2.9963 (0.1406)
$\boldsymbol{\beta_1^U}$	3.0753 (0.4046)	3.0408 (0.2131)	3.0264 (0.1443)	3.1754 (0.4013)	3.2284 (0.2121)	3.2353 (0.1450)
$\boldsymbol{\beta_2^L}$	1.9101	1.8639	1.8274	1.7262	1.7185	1.7160

	(0.1108)	(0.0830)	(0.0512)	(0.1949)	(0.1011)	(0.0682)
$\boldsymbol{\beta_2^c}$	1.9977 (0.0806)	2.0006 (0.0392)	2.0001 (0.0278)	2.0182 (0.0838)	2.0044 (0.0396)	2.0014 (0.0279)
$\boldsymbol{\beta_2^U}$	2.0853 (0.1097)	2.1373 (0.0762)	2.1729 (0.0479)	2.2187 (0.0858)	2.2104 (0.0407)	2.2085 (0.0286)
$\boldsymbol{\beta_3^L}$	0.9753 (0.0495)	0.9864 (0.0303)	0.9971 (0.0188)	0.9849 (0.0487)	0.9778 (0.0247)	0.9773 (0.0169)
$\boldsymbol{\beta_3^c}$	1.0000 (0.0396)	0.9996 (0.0200)	1.0001 (0.0137)	1.0061 (0.0404)	1.0007 (0.0200)	1.0005 (0.0137)
$\boldsymbol{\beta_3^U}$	1.0248 (0.0509)	1.0128 (0.0325)	1.0031 (0.0187)	1.0207 (0.0415)	1.0151 (0.0206)	1.0147 (0.0140)
RMSE	1.8463 (0.4530)	1.4584 (0.1411)	1.3708 (0.1035)	1.3199 (0.5729)	0.7453 (0.2157)	0.5836 (0.1430)

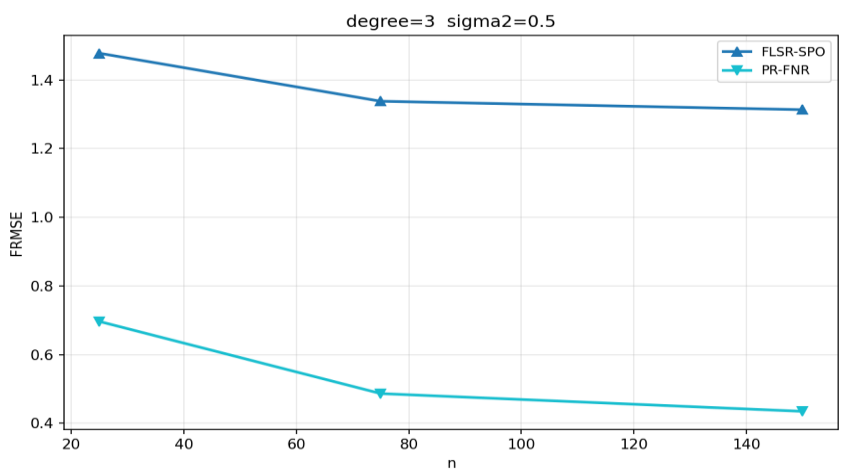

Fig 1. RMSE versus sample size n for the degree-3 model with $\sigma^2 = 0.5$

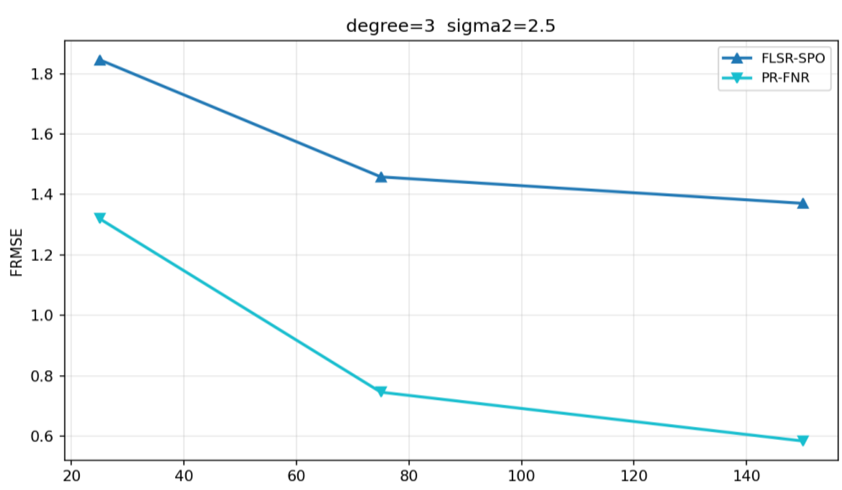

Fig 2. RMSE versus sample size n for the degree-3 model with $\sigma^2 = 2.5$

5.2 Degree-4 Model

The degree-4 results are shown in Tables 4 and 5. The larger standard deviations of the high-order spread coefficients indicate that the design matrix is significantly more ill-conditioned due to the added quartic term. However, the irregular fit of the FLSR-SPO results in a root mean square error exceeding 2.2 in all cases, while the PR-FNR stays below 1.2 in all configurations except the smallest and noisiest. Figures 3 and 4 show that the PR-FNR performance is better in the third-degree case.

As its RMSE values in the 4-order model are larger than those in the 3-order model, the PR-FNR method performs better in the 3-order case, as Figures 3 and 4 show.

Figures 3 and 4 show that the PR-FNR performance is better in the third-degree case.

Table 4. RMSE and average fuzzy-coefficient estimations for the degree-4 model with $\sigma^2 = 0.5$

Parameter	**FLSR-SPO**			**PR-FNR**		
	n = 25	**n = 75**	**n = 150**	**n = 25**	**n = 75**	**n = 150**
$\boldsymbol{\beta_0^L}$	4.6426 (0.5899)	4.7852 (0.3359)	4.8390 (0.1450)	3.9916 (0.4210)	4.0706 (0.2231)	4.0757 (0.1602)
$\boldsymbol{\beta_0^c}$	5.0027 (0.2979)	5.0022 (0.1602)	5.0045 (0.1080)	4.9111 (0.3325)	4.9895 (0.1623)	4.9986 (0.1084)
$\boldsymbol{\beta_0^U}$	5.3629 (0.6004)	5.2192 (0.3546)	5.1699 (0.1510)	5.1582 (0.3460)	5.2367 (0.1703)	5.2459 (0.1138)
$\boldsymbol{\beta_1^L}$	2.9468 (0.1808)	2.9451 (0.1035)	2.9451 (0.0785)	4.2528 (1.0856)	4.5425 (0.5914)	4.6667 (0.4264)
$\boldsymbol{\beta_1^c}$	2.9842 (0.1781)	2.9943 (0.0950)	3.0006 (0.0634)	2.9728 (0.1821)	2.9936 (0.0945)	2.9987 (0.0628)
$\boldsymbol{\beta_1^U}$	3.0215 (0.1902)	3.0435 (0.1052)	3.0562 (0.0758)	3.1187 (0.1907)	3.1395 (0.0993)	3.1447 (0.0659)
$\boldsymbol{\beta_2^L}$	1.9439 (0.1489)	1.9504 (0.0916)	1.9580 (0.0647)	0.9138 (1.1549)	0.5489 (0.6172)	0.3911 (0.4337)
$\boldsymbol{\beta_2^c}$	1.9942 (0.1191)	1.9991 (0.0636)	1.9994 (0.0436)	2.0225 (0.1247)	2.0023 (0.0637)	2.0009 (0.0437)
$\boldsymbol{\beta_2^U}$	2.0445 (0.1475)	2.0480 (0.0946)	2.0408 (0.0657)	2.1195 (0.1302)	2.0993 (0.0669)	2.0979 (0.0458)
$\boldsymbol{\beta_3^L}$	0.9828 (0.0384)	0.9714 (0.0348)	0.9663 (0.0343)	1.2367 (0.4416)	1.3706 (0.2314)	1.4328 (0.1600)
$\boldsymbol{\beta_3^c}$	1.0022 (0.0212)	1.0009 (0.0092)	0.9999 (0.0061)	1.0027 (0.0216)	1.0009 (0.0091)	1.0001 (0.0061)
$\boldsymbol{\beta_3^U}$	1.0216 (0.0363)	1.0304 (0.0337)	1.0335 (0.0343)	1.0507 (0.0227)	1.0490 (0.0096)	1.0481 (0.0064)
$\boldsymbol{\beta_4^L}$	0.4831 (0.0197)	0.4813 (0.0141)	0.4807 (0.0122)	0.4451 (0.0581)	0.4294 (0.0288)	0.4217 (0.0197)
$\boldsymbol{\beta_4^c}$	0.5006 (0.0093)	0.5000 (0.0045)	0.5000 (0.0031)	0.4987 (0.0095)	0.4998 (0.0044)	0.5000 (0.0031)
$\boldsymbol{\beta_4^U}$	0.5181 (0.0182)	0.5187 (0.0139)	0.5194 (0.0123)	0.5237 (0.0100)	0.5248 (0.0047)	0.5250 (0.0033)
RMSE	2.6796 (0.6557)	2.3210 (0.3894)	2.2034 (0.1985)	1.1030 (0.4518)	0.8516 (0.1377)	0.8237 (0.1251)

Table 5. RMSE and average fuzzy-coefficient estimations for the degree-4 model with σ² = 2.5

Parameter	FLSR-SPO			PR-FNR		
	n = 25	n = 75	n = 150	n = 25	n = 75	n = 150
β_0^L	4.6784 (0.7724)	4.7850 (0.4622)	4.8301 (0.2588)	3.8085 (0.7844)	4.0217 (0.3883)	4.0472 (0.2727)
β_0^c	4.9949 (0.6491)	5.0057 (0.3488)	4.9996 (0.2455)	4.7155 (0.7767)	4.9502 (0.3578)	4.9793 (0.2465)
β_0^U	5.3113 (0.8302)	5.2265 (0.4762)	5.1692 (0.2738)	4.9624 (0.8038)	5.1977 (0.3747)	5.2265 (0.2586)
β_1^L	2.9428 (0.4063)	2.9448 (0.2030)	2.9432 (0.1538)	4.1506 (1.1107)	4.5632 (0.6431)	4.7164 (0.4145)
β_1^c	2.9882 (0.4039)	2.9935 (0.2015)	3.0006 (0.1484)	2.9098 (0.4279)	2.9835 (0.2004)	2.9951 (0.1481)
β_1^U	3.0336 (0.4140)	3.0421 (0.2090)	3.0581 (0.1548)	3.0556 (0.4466)	3.1293 (0.2104)	3.1410 (0.1554)
β_2^L	1.9466 (0.2917)	1.9451 (0.1450)	1.9570 (0.1045)	1.0060 (1.1683)	0.5277 (0.6670)	0.3332 (0.4125)
β_2^c	2.0004 (0.2802)	1.9946 (0.1344)	1.9987 (0.0959)	2.0743 (0.2976)	2.0093 (0.1349)	2.0041 (0.0958)
β_2^U	2.0542 (0.2966)	2.0440 (0.1554)	2.0403 (0.1089)	2.1715 (0.3106)	2.1061 (0.1415)	2.1010 (0.1005)
β_3^L	0.9811 (0.0552)	0.9701 (0.0403)	0.9661 (0.0370)	1.2270 (0.4288)	1.3811 (0.2465)	1.4571 (0.1506)
β_3^c	1.0008 (0.0446)	1.0007 (0.0200)	1.0001 (0.0143)	1.0080 (0.0467)	1.0016 (0.0199)	1.0005 (0.0143)
β_3^U	1.0205 (0.0533)	1.0315 (0.0388)	1.0340 (0.0360)	1.0560 (0.0488)	1.0496 (0.0209)	1.0486 (0.0150)
β_4^L	0.4821 (0.0278)	0.4825 (0.0165)	0.4810 (0.0135)	0.4444 (0.0592)	0.4280 (0.0315)	0.4184 (0.0193)
β_4^c	0.5004 (0.0221)	0.5004 (0.0095)	0.5000 (0.0068)	0.4961 (0.0228)	0.4996 (0.0095)	0.4997 (0.0067)
β_4^U	0.5188 (0.0270)	0.5183 (0.0165)	0.5189 (0.0134)	0.5212 (0.0239)	0.5246 (0.0100)	0.5247 (0.0071)
RMSE	2.9773 (0.7977)	2.4149 (0.3865)	2.2625 (0.2146)	1.7853 (0.8177)	1.0657 (0.2361)	0.9351 (0.1680)

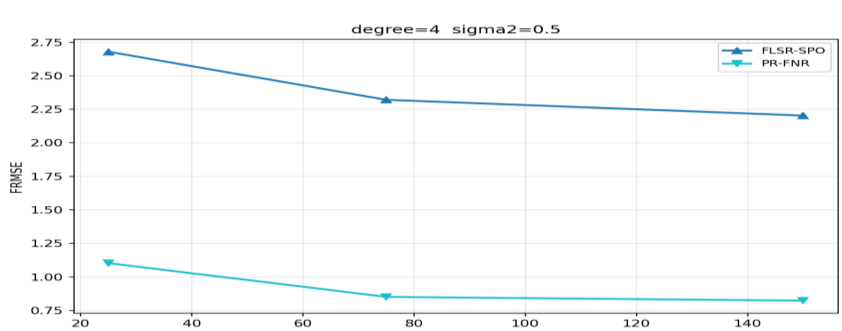

Fig 3. RMSE versus sample size n for the degree-4 model with σ² = 0.5

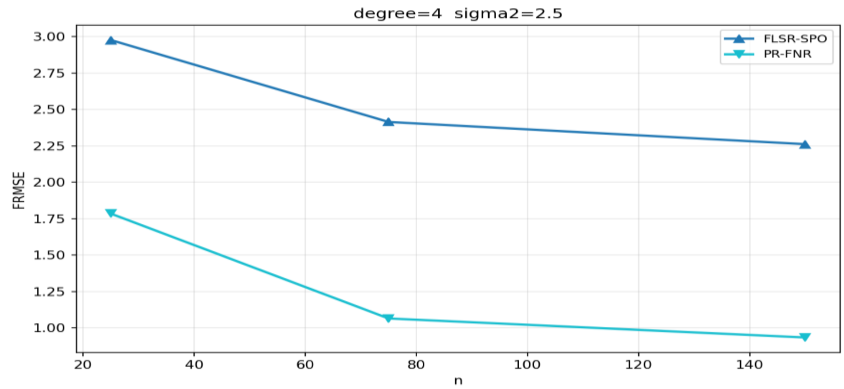

Fig 4. RMSE versus sample size n for the degree-4 model with $\sigma^2 = 2.5$

5.3 The Aggregate Comparison

The root mean square error (RMSE) of the two estimators for each of the twelve case studies is summarized in Table 6. The consistency of fuzzy regression in determining the RMSE is demonstrated by the fact that the error rises with variance and decreases with sample size, confirming the superiority of the PR-FNR method.

Table 6. Averaged RMSE of the two estimators across all twelve cases

k	σ^2	FLSR-SPO			PR-FNR		
		25	75	150	25	75	150
3	0.5	1.4778	1.3380	1.3134	0.6963	0.4863	0.4345
	2.5	1.8463	1.4584	1.3708	1.3199	0.7453	0.5836
4	0.5	2.6796	2.3210	2.2034	1.1030	0.8516	0.8237
	2.5	2.9773	2.4149	2.2625	1.7853	1.0657	0.9351

6 Conclusions and Future Works

First, with almost convergent dispersion for both methods and all cases studied at the fuzzy response center, the simulation results show that the center coefficients are quite close to their true values. Therefore, the fuzzy nonlinear regression problem is clearly defined for the center regardless of the estimator chosen, the spread is the only major difficulty.

Second, the ridge penalty is essential for estimating the spreads. The RMSE, which aggregates the squared errors of the center, upper, and lower values, increases proportionately to the increasing ill-conditioning of the design matrix at higher degrees as FLSR-SPO fits the spreads using an unregularized least-squares criterion. Instead, PR-FNR minimizes this variance and keeps the fitted fuzzy edges near the truth by reducing the high-order coefficients using the penalty term and adjusting the degree of reduction using cross-validation. The related cost is a slight bias that may be seen in a few high-order spread coefficients at degree four; nevertheless, the variance reduction significantly exceeds this bias, resulting in a significant decrease in the RMSE

Third, when the estimate problem gets more difficult, the PR-FNR method becomes more advantageous. Its most significant advantage is shown when employing high-score polynomial models, and this benefit is more obvious when the sample size is small, since the combination of data limitation and inadequate design matrix adaptation makes parameter estimation more difficult and emphasizes the method's ability to produce more accurate and stable estimates.

For future work, it is advisable to use this method to nonlinear regression models that do not depend on polynomials, including exponential and logarithmic models, and to extend its evaluation to higher-order models.

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