Journal of Iraqi Al-Khwarizmi (JIKh) Volume: 10 Issue: 2 Year: 2026 pages: 223-229

On the Connected Subgraphs Arrangement of the Wheel Graph : Intersection Lattice, Supersolvability, and Freeness

Authors Names

[a]*Rabeaa Gataa Abtan*

[b]*Hassan Ahmed Ghanim*

Publication date: 13 /8 /2026

Keywords: Wheel Graph; Connected Subgraphs Arrangement; Hyperplane Arrangements; Intersection Lattice; Supersolvability; Freeness

ABSTRACT

This paper presents a comprehensive analysis of the connected subgraphs arrangement $A_{W_5}^{conn}$ associated with the wheel graph W_5. Following the definition by [1], this arrangement consists of hyperplanes $\ker(\sum_{i\in I} x_i)$ where $I \subseteq \{0,1,2,3,4\}$ induces a connected subgraph of W_5 and either contains the central vertex 0 or forms a connected interval on the cycle C_4. We explicitly compute the full intersection lattice of this arrangement, contains all 1570 non-empty intersections. Using this lattice , we provide two distinct proofs of fundamental algebraic properties. First, we demonstrate that $A_{W_5}^{conn}$ is not supersolvable by exhibiting a specific element that fails to be modular. Second, using a localization argument centered on a non-chordal cyclic subarrangement, we apply Stanley's theorem to prove that $A_{W_5}^{conn}$ is not free. Finally, we extend our discussion to the general family $A_{W_n}^{conn}$ for $n \geq 5$, conjecturing that these connected subgraphs arrangements are neither supersolvable nor free for all $n \geq 5$.

1. Introduction

The study of hyperplane arrangements from graphs has a rich history in algebraic combinatorics. While graphic arrangements $A_G = \{x_i - x_j = 0 \mid \{i,j\} \in E(G)\}$ are well known, a more recent family is that of connected subgraphs arrangements. For a given graph $G = (V, E)$ consists of a finite set V of vertices and a set E of edges, the connected subgraphs arrangement $\mathcal{A}_G^{conn}$ has hyperplanes of the form $\sum_{i\in I} x_i = 0$ where $I \subseteq V(G)$ gives a connected subgraph of G. This paper focuses on the connected subgraphs arrangement of the wheel graph W_5. The wheel graph W_n has a central vertex 0 connected to all vertices of a cycle C_{n-1}. This family is a good place to test combinatorial and algebraic properties because it is symmetric but not simple.
To work with this, we use some basic definitions. Let $\mathcal{A}$ be an arrangement and $L(\mathcal{A})$ includes V as the intersection of the empty collection of hyperplanes. Define a partial order relation on $L(\mathcal{A})$ by $X \leq Y$ if and only if $Y \subseteq X$ for each $X, Y \in L(\mathcal{A})$. The poset of $\mathcal{A}$ is called the lattice of $\mathcal{A}$ and denoted by $L_{\mathcal{A}}$. For $X \in L(\mathcal{A})$ define the localization A_X of A by $A_X = \{H \in A | X \subseteq H\}$[2]. An arrangement $\mathcal{A}$ is supersolvable if its intersection lattice $L(\mathcal{A})$ contains a maximal chain of modular elements [2]. An arrangement A in R^n is free if its module of derivations is a free module of rank n over the polynomial ring [2]. The projective dimension pd(M), of a finitely generated graded S-module M is defined as the length k of a minimal free graded resolution of M. This resolution is given by the sequence: $0 \to F_k \to \ldots \to F_1 \to F_0 \to M \to 0$ where Fi are free graded S modules [6].
Our main contribution is a complete computational analysis of $A_{W_5}^{conn}$, conducted via its full intersection lattice , which reveals that the arrangement $A_{W_5}^{conn}$ is neither supersolvable nor free, then generalize these results by introducing the family $A_{W_n}^{conn}$ and discussing the broader implications of our findings

2. Preliminaries and Definitions

Definition 2.1: The wheel graph W_5 consists of a central vertex 0 connected to all vertices of a 4-cycle. The cycle vertices are labeled 1,2,3,4. The edge set is

---

[a] *Mustansiriyah University, College of Science, Department of Mathematics, Baghdad, Iraq.*
E-Mail: dr.rabeea @ uomustansiriyah.edu.iq
[b] *Mustansiriyah University, College of Science, Department of Mathematics, Baghdad, Iraq.*
E-Mail: hassanahmed@uomustansiriyahedu.iq

$$E(W_5) = \{(1,2),(2,3),(3,4),(4,1),(0,1),(0,2),(0,3),(0,4)\}$$

Definition 2.2: Following the definition in [1], the connected subgraphs arrangement $A_{W_5}^{conn}$ in R^5 with coordinates$(x_0, x_1, x_2, x_3, x_4)$is:

$A_{W_5}^{conn} = \{ ker(\sum_{i \in I} x_i) \mid I \subseteq \{0,1,2,3,4\}, I \text{ induces a connected subgraph of } W_5\}$.

More explicitly, a hyperplane $\sum_{i \in I} x_i = 0$ is included if either:

1. $0 \in I$ (any connected subgraph containing the central vertex), or
2. $I \subseteq \{1,2,3,4\}$ forms a connected interval on the cycle C_4 .

This yields the following 29 hyperplanes, which we label H_1 through H_{29}

Table (3.1): The 29 hyperplanes of $\boldsymbol{A_{W_5}^{conn}}$.

Hyperplanes Containing Vertex 0		**Connected Intervals on the Cycle C_4**		
Hyperplane	**Equation**	**Hyperplane**	**Equation**	**Type**
H_1	$x_0 = 0$	H_{17}	$x_1 = 0$	Singleton
H_2	$x_0 + x_1 = 0$	H_{18}	$x_2 = 0$	Singleton
H_3	$x_0 + x_2 = 0$	H_{19}	$x_3 = 0$	Singleton
H_4	$x_0 + x_3 = 0$	H_{20}	$x_4 = 0$	Singleton
H_5	$x_0 + x_4 = 0$	H_{21}	$x_1 + x_2 = 0$	Edge
H_6	$x_0 + x_1 + x_2 = 0$	H_{22}	$x_2 + x_3 = 0$	Edge
H_7	$x_0 + x_1 + x_3 = 0$	H_{23}	$x_3 + x_4 = 0$	Edge
H_8	$x_0 + x_1 + x_4 = 0$	H_{24}	$x_1 + x_4 = 0$	Edge
H_9	$x_0 + x_2 + x_3 = 0$	H_{25}	$x_1 + x_2 + x_3 = 0$	Path
H_{10}	$x_0 + x_2 + x_4 = 0$	H_{26}	$x_2 + x_3 + x_4 = 0$	Path
H_{11}	$x_0 + x_3 + x_4 = 0$	H_{27}	$x_1 + x_3 + x_4 = 0$	Path
H_{12}	$x_0 + x_1 + x= + x_3 = 0$	H_{28}	$x_1 + x_2 + x_4 = 0$	Path
H_{13}	$x_0 + x_1 + x_2 + x_4 = 0$	H_{29}	$x_1 + x_2 + x_3 + x_4 = 0$	Entire cycle
H_{14}	$x_0 + x_1 + x_3 + x_4 = 0$			
H_{15}	$x_0 + x_2 + x_3 + x_4= = 0$			
H_{16}	$x_0 + x_1 + x_2 + x_3 + x_4 = 0$			

3. The Intersection Lattice of

Using Python with SymPy, we counted all the kinds of intersections for $L(\boldsymbol{A}_{W_5}^{conn})$. The lattice was built step by step: began with the space R^5, then we got more intersections by adding hyperplane. The lattice is ordered by rank (which is the codimension) of the intersections. Table 2 show how many elements in each rank.

Table 2: The intersection lattice $\boldsymbol{L}(\boldsymbol{A}_{W_5}^{conn})$ of $\mathbf{A}_{\mathbf{W}_5}^{\mathbf{conn}}$

Rank	**Dimension**	**Total Elements**	**Example Intersection**
0	5	1	Universe: R^5
1	4	29	$H_1 : x_0 = 0$
2	3	256	$H_1 \cap H_7 : x_0 = 0 , x_1 = -x_3$
3	2	743	$H_2 \cap H_3 \cap H_4 : x_0 = -x_3 , x_1 = x_3 , x_2 = x_3$
4	1	541	$H_2 \cap H_3 \cap H_4 \cap H_5 : x_0 = -x_4 , x_1 = x_4 , x_2 = x_4 , x_3 = x_4$
5	0	1	Origin: $x_0 = x_1 = x_2 = x_3 = x_4 = 0$
Total		**1571**	

The lattice exhibits rich structure. For instance, intersections of two hyperplanes at rank 2 often involve three defining equations, as the system of two linear equations in R^5 defines a 3-dimensional subspace.

4. $\mathbf{A}_{\mathbf{W}_5}^{\mathbf{conn}}$ is not Supersolvable

In this section we will prove that $\mathbf{A}_{\mathbf{W}_5}^{\mathbf{conn}}$ is not Supersolvable

Corollary 4.1 [3]: An element $X \in L$ is modular if and only if $X + Y \in L$ for all $Y \in L$.

Proposition 4.2: $\mathbf{A}_{\mathbf{W}_5}^{\mathbf{conn}}$ is not supersolvable.

Proof: The intersections lattice $\boldsymbol{L}(\boldsymbol{A}_{W_5}^{conn})$ of $\boldsymbol{A}_{W_5}^{conn}$ has no modular element of rank(3) for example $\boldsymbol{X} \in \boldsymbol{L}(\boldsymbol{A}_{W_5}^{conn})$ s.t $\boldsymbol{X} = \{H_1 \cap H_2 \cap H_3 \cap H_6 \cap H_{17} \cap H_{18} \cap H_{21}\}$

$= \{(x_0, x_1, x_2, x_3, x_4) \in R^5 | x_0 = 0, x_1 = 0, x_2 = 0\}$ is not modular since $\exists\, \boldsymbol{Y} \in \boldsymbol{L}(\boldsymbol{A}_{W_5}^{conn})$

$\mathrm{Y} = \{H_1 \cap H_7 \cap H_9 \cap H_{22}\} = \{(x_0, x_1, x_2, x_3, x_4) \in R^5 | x_0 = 0, x_1 = -x_3, x_2 = -x_3\}$

And $\boldsymbol{X} + \boldsymbol{Y} = \{(x_0, x_1, x_2, x_3, x_4) \in R^5 | x_0 = 0, x_1 = x_2\} \notin \boldsymbol{L}(\boldsymbol{A}_{W_5}^{conn})$,

Therefore by Corollary(5.1) $\mathbf{A}_{\mathbf{W}_5}^{\mathbf{conn}}$ is not Supersolvable .

5. $\mathbf{A}_{\mathbf{W_5}}^{\mathbf{conn}}$ is not Free

In this section we will prove that $\mathbf{A}_{\mathbf{W_5}}^{\mathbf{conn}}$ is not free arrangement. The proof uses a localization argument and relies on a classic theorem by Stanley connecting graphic arrangements to chordal graphs [1]

Theorem 5.1 [4]. $D(\mathcal{A}_G)$ is free module if and only if the graph G is chordal (i.e., has no induced cycles of length ≥ 4). Therefore $pd(D(A_G)) = 0$

Proposition 5.2 [5]: Let $X \in L(A)$. Then $pd(A_X) \le pd(A)$

Proposition 5.3. The arrangement $\boldsymbol{A_{W_5}^{conn}}$ is not free

Proof : Consider

$X = \{x_1 = x_2 = x_3 = x_4 = 0\} \in \boldsymbol{L}(\boldsymbol{A_{W_5}^{conn}})$

The localization A_X consists of all hyperplanes whose defining linear forms vanish identically on, that is

$$\mathrm{A_X} = \{(\mathrm{x}_1), (\mathrm{x}_2), (\mathrm{x}_3), (\mathrm{x}_4), (\mathrm{x}_1+\mathrm{x}_2), (\mathrm{x}_2+\mathrm{x}_3), (\mathrm{x}_3+\mathrm{x}_4), (\mathrm{x}_4+\mathrm{x}_1), (\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_3), (\mathrm{x}_2+\mathrm{x}_3 + \mathrm{x}_4), (\mathrm{x}_3+\mathrm{x}_4+\mathrm{x}_1), (\mathrm{x}_4+\mathrm{x}_1+\mathrm{x}_2), (\mathrm{x}_1+\mathrm{x}_2+\mathrm{x}_3+\mathrm{x}_4)\}$$

Thus, $\mathrm{A_X}$contains the hyperplanes

$\mathrm{x}_1+\mathrm{x}_2 = 0, \quad \mathrm{x}_2+\mathrm{x}_3 = 0, \quad \mathrm{x}_3+\mathrm{x}_4 = 0, \quad \mathrm{x}_4+\mathrm{x}_1 = 0$

Which form a subarrangement corresponding to a cycle of length four. This cycle is chordless and hence defines a non-chordal graph. By Stanley's Theorem (6.1), a graphic arrangement is free if and only if the underlying graph is chordal. Therefore, the above subarrangement is not free, which obstructs the freeness of the localized arrangement $\mathrm{A_X}$. Consequently,

$pd(D(A\_X)) \ge 1$. Since projective dimension does not decrease under localization, we conclude that $pd(D(A)) \ge pd(D(A_X)) \ge 1$

∎

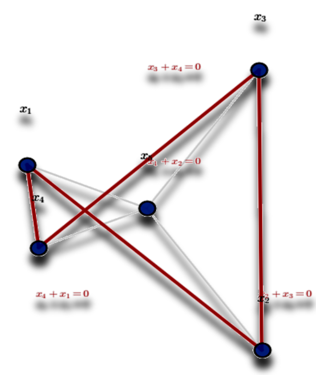

Figure 1 :The W_5 arrangement

Remark 5.4: in Figure 1 blue spheres represent variables x_1, x_2, x_3, x_4. Thick red edges represent the C4 cycle resulting from the substitution $x_1 = x_2 = x_3 = x_4 = 0$, satisfying: $x_1 + x_2 = 0$ $x_2 + x_3 = 0$ $x_3 + x_4 = 0$, $x_4 + x_1 = 0$ This is a chordless C4 cycle. By Stanley's theorem, its existence prevents the original arrangement $\boldsymbol{A_{W_5}^{conn}}$ from being free. Dashed gray edges (connected to x_0) disappear under localization.

Remark 5.5: This proof is significant because $\boldsymbol{A_{W_5}^{conn}}$ is not itself a graphic arrangement, but its non-freeness is establishes by the presence of a non-chordal graphic arrangement within one of its localizations.

6. Generalization to the Family

The results for $\boldsymbol{W_5}$ suggest a natural generalization to all wheel graphs $\boldsymbol{W_n}$ for n ≥ 4. We define the connected subgraphs arrangement $\boldsymbol{A_{W_n}^{conn}}$ in R^n with coordinates $(x_0, x_1, x_2, .., x_{n-1})$ as:

$$A_{W_n}^{conn} = \{ ker(\textstyle\sum_{i \in I} x_i) \mid I \subseteq \{0,1, .., n-1\}, I \textit{ induces a connected subgraph of } W_n\}$$

Equivalently, a hyperplane is included if either $0 \in I$ (any connected subgraph containing the central vertex) or $I \subseteq \{1, \dots, n-1\}$ forms a connected interval on the cycle C_{n-1}

6.1. Intersection Lattice $L(A_{W_n}^{conn})$ of $A_{W_n}^{conn}$

The intersection lattice $\boldsymbol{L(A_{W_n}^{conn})}$ of $\boldsymbol{A_{W_n}^{conn}}$will be significantly more complex than that of W_5. However, its structure can be analyzed recursively. The element $Z_n = \{x_1 = x_2 = \cdots = x_4 = 0\}$ (the x_0-axis) will always be present in the lattice. The localization of $\boldsymbol{A_{W_n}^{conn}}$ at Z_n yields an arrangement in R^{n-1}(on coordinates $x_1, \dots, x_{n-1}$) consisting of all hyperplanes whose forms are independent of x_0: $(\boldsymbol{A_{W_n}^{conn}})_{Z_n} \sim = \{$connected intervals on C_{n-1}. A subarrangement within this localization is the graphic arrangement of the cycle C_{n-1}itself, given by the forms $\{x_i + x_{i+1} = 0\}$ for i = 1,...,n − 2 and $x_{n-1} + x_1 = 0$ (with cyclic indices (identifying x_n with x_1).

6.2. Non-Freeness for n ≥ 5

(7.2)Proposition. The arrangement $\boldsymbol{A}_{W_{\mathbf{n}}}^{conn}$ is not free for n ≥ 5

Proof: Let $Z_n = \{x_1 = x_2 = \cdots = x_4 = 0\}$ Consider the localization $(\boldsymbol{A}_{W_{\mathbf{n}}}^{conn})_{Z_n}$. This localization has the graphic arrangement of the cycle C_{n-1}. given by: $x_\mathrm{i} + x_{i+1} = 0, i = 1, \dots, n - 2, and\, x_{n-1} + x_1 = 0$

For n ≥ 5, we have n − 1 ≥ 4, so C_{n-1} is non-chordal. By Stanley's Theorem(6.1), the graphic arrangement is not free. So, the localization is not free. This shows that$\boldsymbol{A}_{W_{\mathbf{n}}}^{conn}$ is not free. This argument holds for all n ≥ 5.

■

6.3. Remark (Non-Supersolvability Argument for n ≥ 5)

The non-supersolvability argument for W_5 used a specific pair of elements (X and Y) whose join was not in the lattice. This pattern is expected to extend. For W_n , one can consider the element $X_n = \{x_0 = 0, x_1 = 0, x_2 = 0\}$ (The intersection of H_1, H_{17}, H_{18} and several other extra hyperplanes) and another element $Y_n = \{x_0 = 0, x_1 = -x_3, x_2 = -x_3\}$. The join $X_n \vee Y_n$would likely give the subspace $\{x_1 = 0 , x_1 = x_2\}$, which is not an element of the lattice for $n \geq 5$ because it cannot be written as an intersection of the defining hyperplanes of $\boldsymbol{A}_{W_{\mathbf{n}}}^{conn}$. The existence of such a pair would violate the modularity condition, showing non-supersolvability. A full proof would need a more detailed lattice analysis, but the structural similarity to the n = 5 case strongly shows the property holds for all larger n

7. Conclusions

We have done a full investigation of the connected subgraphs hyperplane arrangement $\boldsymbol{A}_{W_5}^{conn}$ with the wheel graph $\boldsymbol{W_5}$. By fully listing its intersection lattice, we gave concrete data that let two independent algebraic proofs. We showed $\boldsymbol{A}_{W_5}^{conn}$ is neither supersolvable nor free. The non-freeness proof, in particular, shows a strong way: finding non-freeness in a non-graphic arrangement by using a localization that gives a non-chordal graphic subarrangement. Finally, we generalized these results to the infinite family $\boldsymbol{A}_{W_5}^{conn}$, guessing that they share these same non-properties for all n ≥ 5. This work shows the tight connection between graph theory and the algebraic combinatorics of hyperplane arrangements.

REFERENCES

[1] L. Giordani, T. Möller, P. Mücksch, and G. Röhrle, "On connected subgraph arrangements," Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry, 2026. doi: 10.1007/s13366-026-00858-6.

[2] R. G. A. Al-Aleyawee and A. A. A. Abdullah, "Some new result on Shi arrangement," AIP Conference Proceedings, 2023.

[3] R. Al-Aleyawee, Hypersolvable and Free Arrangements, Ph.D. thesis, College of Science, Mustansiriyah University, 2005.

[4] P. H. Edelman and V. Reiner, "Free hyperplane arrangements between A_{n-1} and B_n," Mathematische Zeitschrift, vol. 215, no. 3, pp. 347–365, 1994.

[5] L. Mühlherr, A Graph Theoretical Perspective on Hyperplane Arrangement Theory, Ph.D. dissertation, Universität Bielefeld, 2025.

[6] T. Abe, L. Kühne, P. Mücksch, and L. Mühlherr, "Projective dimension of weakly chordal graphic arrangements," Algebraic Combinatorics, vol. 8, no. 1, pp. 157–174, 2025.
[7] R. G. A. Al-Aleyawee and R. S. Haraj, "Hyperfactored of the arrangements A(G24) and A(G27)," Journal of Physics: Conference Series, vol. 1818, no. 1, Art. 012144, 2021.
[8] P. Orlik and H. Terao, Arrangements of Hyperplanes, Grundlehren der mathematischen Wissenschaften, vol. 300, Springer-Verlag, Berlin, 1992.
[9] R. P. Stanley, "Supersolvable lattices," Algebra Universalis, vol. 2, pp. 197–217, 1972.
[10] H. Terao, "Modular elements of lattices and topological fibration," Advances in Mathematics, vol. 62, pp. 135–154, 1986. doi: 10.1016/0001-8708(86)90097-6.
[11] T. Hoge and G. Röhrle, "On supersolvable reflection arrangements," Journal of Algebraic Combinatorics, vol. 38, pp. 415–436, 2013.
[12] G. Denham, M. Garrousian, and S. Tohaneanu, "Modular decomposition of the Orlik–Terao algebra of a hyperplane arrangement," Annals of Combinatorics, vol. 18, no. 2, pp. 289–312, 2014.
[13] S. Tsujie, "Modular construction of free hyperplane arrangements," SIGMA, vol. 16, Art. 080, 2020. doi: 10.3842/SIGMA.2020.080.
[14] T. N. Tran and S. Tsujie, "MAT-free graphic arrangements and a characterization of strongly chordal graphs by edge-labeling," Algebraic Combinatorics, vol. 6, no. 6, pp. 1447–1467, 2023. doi: 10.5802/alco.319.
[15] P. Mücksch, G. Röhrle, and S. Wiesner, "Free multiderivations of connected subgraph arrangements," Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, vol. 30, 2026. doi: 10.2422/2036-2145.202407_003.
[16] L. Giordani, P. Mücksch, G. Röhrle, and J. Schmitt, "Hyperpolygonal arrangements," Glasgow Mathematical Journal, published online 14 July 2025.
[17] T. Hoge and G. Röhrle, "Addition-deletion theorems for factorizations of Orlik–Solomon algebras and nice arrangements," European Journal of Combinatorics, vol. 50, pp. 73–90, 2015.
[18] M. Jambu and H. Terao, "Free arrangements of hyperplanes and supersolvable lattices," Advances in Mathematics, vol. 52, pp. 248–258, 1984.
[19] G. Ziegler, "Matroid representations and free arrangements," Transactions of the American Mathematical Society, vol. 320, no. 2, pp. 525–541, 1990.
[20] K. Saito, "Theory of logarithmic differential forms and logarithmic vector fields," Journal of the Faculty of Science, University of Tokyo, Section IA Mathematics, vol. 27, pp. 265–291, 1980.