Pythagorean Fuzzy Q-Filters of Q-Algebras

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ABSTRACT

In this paper, we introduce the notion of a level Pythagorean fuzzy Q-filter (LPFQ-filter) in a bounded Q-algebra. The proposed structure combines the theory of Q-filters with Pythagorean fuzzy sets by requiring the nonempty (α, β)-level sets of a Pythagorean fuzzy set to be Q-filters. We first establish the basic framework of LPFQ-filters and provide a non-constant example to demonstrate that the proposed notion is nontrivial. We then investigate the structural properties of their level sets. In particular, we prove a nestedness property and characterize the intersection of two (α, β)-level sets in terms of the maximum membership threshold and the minimum non-membership threshold. As a consequence, we show that the intersection of two nonempty level sets of an LPFQ-filter is again a Q-filter. These results establish a direct connection between Pythagorean fuzzy structures and ordinary Q-filters and provide a framework for further investigations of fuzzy filter structures in bounded Q-algebras.

Keywords: Q-algebra; Q-filter; Pythagorean fuzzy set; Pythagorean fuzzy Q-filter; level sets; fuzzy algebra.

1. Introduction

Q-algebras constitute an important class of non-classical algebraic structures and have been studied in connection with several related systems, including BCK- and BCI-algebras. The concept of a Q-algebra was introduced by Neggers et al. as a generalization of several algebraic structures [10]. Since then, various algebraic properties and filter-type structures associated with Q-algebras have been investigated.

The theory of fuzzy sets, introduced by Zadeh [12], provides a natural framework for representing gradual membership and uncertainty. Subsequently, Atanassov introduced intuitionistic fuzzy sets, in which membership and non-membership degrees are considered simultaneously [2,3]. These generalized fuzzy structures have been extensively investigated in different algebraic settings. In particular, fuzzy and intuitionistic fuzzy filter structures have been studied in Q-algebras and related algebraic systems [9,11].

Pythagorean fuzzy sets, introduced by Yager [13,14], extend the intuitionistic fuzzy framework by allowing the membership and non-membership degrees to satisfy

$$\mu_p(\xi)^2 + \nu_p(\xi)^2 \leq 1, \quad \xi \in T.$$

This condition provides a larger admissible domain for membership and non-membership degrees and has made Pythagorean fuzzy sets useful in the representation of uncertain and imprecise information.

Motivated by the interaction between fuzzy structures and algebraic filters, this paper introduces the notion of a level Pythagorean fuzzy Q-filter, abbreviated as an LPFQ-filter, in a bounded Q-algebra. The proposed notion is formulated through the (α, β)-level sets of a Pythagorean fuzzy set. More precisely, a Pythagorean fuzzy set is called an LPFQ-filter when each nonempty admissible (α, β)-level set is a Q-filter of the underlying bounded Q-algebra.

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The main purpose of this approach is to establish a direct connection between the Pythagorean fuzzy description and the underlying ordinary Q-filter structure. We first introduce the proposed LPFQ-filter notion and provide a non-constant example demonstrating that the definition is nontrivial. We then investigate the structural behavior of the associated level sets.

In particular, we establish a nestedness property showing how (α, β)-level sets vary with their membership and non-membership thresholds. We further characterize the intersection of two level sets by $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2) = C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\})$.

As a consequence, whenever this intersection is nonempty, it is again a Q-filter of the underlying bounded Q-algebra. These results clarify the relationship between LPFQ-filters and ordinary Q-filters through their level-set structure.

The remainder of the work is organized as follows. Section two, consists the basic concepts of bounded Q-algebras, Q-filters, Pythagorean fuzzy sets, and (α, β)-level sets. Section 3 introduces LPFQ-filters and establishes their basic level-set properties, including nestedness and the intersection result. Finally, the conclusions summarize the main findings and indicate possible directions for future research. Tthis section, we recall the basic notions concerning Q-algebras, Q-filters, and Pythagorean fuzzy sets that will be used throughout the paper.

Definition 2.1. [10] Let T be a nonempty set equipped with a binary operation * and a distinguished element 0. The algebraic structure (T, *, 0) is called a Q-algebra if, for all ξ, ζ, ω ∈ T,

$$\xi * \xi = 0, \qquad \text{(Q1)}$$

$$\xi * 0 = \xi, \qquad \text{(Q2)}$$

$$(\xi * \zeta) * \omega = (\xi * \omega) * \zeta. \qquad \text{(Q3)}$$

Definition 2.2. [1] A Q-algebra (T, *, 0) is called bounded if there exists an element e ∈ T such that $\xi \leq e, \forall \xi \in T$.

The element e is said to be the unit of T.

Remark 2.3. *Let (T, *, 0) be a bounded Q-algebra with unit e. For every ξ ∈ T, we writ*

$$e\ \xi^c = e * \xi.$$

Throughout this paper, all Q-algebras are assumed to be bounded and e denotes their unit.

Definition 2.4. Let (T, *, 0) be a bounded Q-algebra with unit e. A nonempty subset F ⊆ T is called a Q-filter of T if e ∈ F, and whenever $(\xi^c * \zeta)^c \in F$ and ζ ∈ F, then ξ ∈ F, for all ξ, ζ ∈ T \ {0}.

Example 2.5. Take T = {0, r, s, t, u} equipped such that a binary operation * given by

*	0	r	s	t	u
0	0	0	0	0	0
r	r	0	r	r	0
s	s	s	0	s	0
t	t	t	t	0	0
u	u	0	0	0	0

Then (T, *, 0) is a bounded Q-algebra with unit e. For each ξ ∈ T, define $\xi^c = u * \xi$.

Hence, $0^c = u$, $r^c = s^c = t^c = u^c = 0$, and therefore $0^{cc} = 0$, $r^{cc} = s^{cc} = t^{cc} = u^{cc} = u$.

Consider the subset F = {r, s, t, u}. Clearly, u ∈ F.

Let $\xi, \zeta \in T \setminus \{0\}$ and suppose that $(\xi^c * \zeta)^c \in F$ and $\zeta \in F$.

Since $\zeta \neq 0$, we have $\zeta^c = 0$. Moreover, for $\xi \neq 0$, $\xi^{cc} = e$. Consequently, $(\xi^c * \zeta)^c = (u * 0)^c = u^c = 0$. Since $0 \notin F$, the antecedent $(\xi^c * \zeta)^c \in F$ cannot hold. Thus the defining implication of a Q-filter is satisfied. Therefore, $F = \{ r, s, t, u \}$ is a Q-filter of T.

Proposition 2.6. Let (T, *, 0) be the bounded Q-algebra considered in Example 2.5. Then each of the following subsets is a Q-filter of T: $\{u\}$, $\{r,u\}$, $\{r,s,u\}$, $\{ r, s, t, u \}$, T. In particular, T admits proper nontrivial Q-filters.

Proof. Let F be one of the sets u}, $\{r,u\}$, $\{r,s,u\}$, $\{ r, s, t, u \}$. Clearly, $u \in F$ and $0 \notin F$. Let $\xi, \zeta \in T \setminus \{0\}$ and suppose that $(\xi^c * \zeta)^c \in F$ and $\zeta \in F$. Since $\zeta \neq 0$, the definition of ^{c} gives $\zeta^c = 0$. Moreover, for every $\xi \neq 0$, $\xi^{cc} = e$. Therefore, $(\xi^c * \zeta)^c = (u * 0)^c = u^c = 0$. But $0 \notin F$, which contradicts the assumption $(\xi^c * \zeta)^c \in F$.

Hence the implication in Definition 2.4 is satisfied, and therefore F is a Q-filter of T. Finally, T is a Q-filter trivially, since it contains e and every element of $T \setminus \{0\}$ belongs to T. Thus, u}, $\{r,u\}$, $\{r,s,u\}$, $\{ r, s, t, u \}$, T are all Q-filters of T. □

Definition 2.7. [13,14] Let T be a nonempty set. A Pythagorean fuzzy set on T is an ordered pair $P = (\mu_p, \nu_p)$, where $\mu_p:T\to[0,1]$ and $\nu_p:T\to[0,1]$ are the membership and non-membership functions, respectively, such that $\mu_p(\xi)^2 + \nu_p(\xi)^2 \leq 1$, $\forall\, \xi \in T$.

Definition 2.8. Let P be a Pythagorean fuzzy set on T. For $\alpha, \beta \in [0,1]$, define $M_p(\alpha) = \{\xi \in T : \mu_p(\xi) \geq \alpha\}$, and $N_p(\beta) = \{\xi \in T : \nu_p(\xi) \leq \beta\}$. Moreover, whenever $\alpha^2 + \beta^2 \leq 1$, the (α, β)-level set of P is defined by $C_p(\alpha,\beta) = \{\xi \in T : \mu_p(\xi) \geq \alpha,\ \nu_p(\xi) \leq \beta\}$.

3. Pythagorean Fuzzy Q-Filters

Definition 3.1. Let (T, *, 0) be a bounded Q-algebra with unit e, and let P be a Pythagorean fuzzy set on T. We call P a level Pythagorean fuzzy Q-filter of T, briefly an LPFQ-filter, if for every $\alpha, \beta \in [0,1]$ satisfying $\alpha^2 + \beta^2 \leq 1$, the nonempty (α, β)-level set $C_p(\alpha,\beta)$ defined in Definition 2.8 is a Q-filter.

Example 3.2. Let $T = \{0,a,b,c,e\}$ be equipped with the binary operation * given by

*	0	a	b	c	e
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	0
e	e	0	0	0	0

Then (T, *, 0) is a bounded Q-algebra with unit e. For each $\xi \in T$, define $\xi^c = e * \xi$. Thus, $0^c = e$, $a^c = b^c = c^c = e^c = 0$, and consequently $0^{cc} = 0$, $a^{cc} = b^{cc} = c^{cc} = e^{cc} = e$.

Define a Pythagorean fuzzy set $P = (\mu_p,\nu_p)$ on T by

ξ	$\mu\_P(\xi)$	$\nu\_P(\xi)$
0	0.20	0.80
a	0.40	0.60
b	0.60	0.40
c	0.70	0.30
e	0.90	0.10

For every $\xi \in T$, $\mu_p(\xi)^2 + \nu_p(\xi)^2 \leq 1$. Indeed,

$$(0.20)^2 + (0.80)^2 = 0.68$$

$$(0.40)^2 + (0.60)^2 = 0.52$$

$$(0.60)^2 + (0.40)^2 = 0.52$$

$$(0.70)^2 + (0.30)^2 = 0.58$$

$$(0.90)^2 + (0.10)^2 = 0.82$$

Hence P is a non-constant Pythagorean fuzzy set. For $\alpha, \beta \in [0,1]$, $\alpha^2 + \beta^2 \leq 1$, consider the (α,β)-level set $C_p(\alpha,\beta) = \{\xi \in T : \mu_p(\xi) \geq \alpha, \nu_p(\xi) \leq \beta\}$.

Since $0.20 < 0.40 < 0.60 < 0.70 < 0.90$

and $0.80 > 0.60 > 0.40 > 0.30 > 0.10$, every nonempty level set is one of $\{e\}$, $\{c,e\}$, $\{b,c,e\}$, $\{a,b,c,e\}$, T.

Now, for every $\xi, \zeta \in T \setminus \{0\}$, we have $\zeta^c = 0$, $\xi^{cc} = e$, and hence $(\xi^c * \zeta)^c = (e * 0)^c = e^c = 0$. Therefore, each proper nonempty level set listed above contains e and does not contain 0, and consequently satisfies the defining condition of a Q-filter. The set T is also a Q-filter. Thus every nonempty (α,β)-level set of P is a Q-filter of T. Hence, by Definition 3.1, P is an LPFQ-filter of T.

Theorem 3.3. Let P be an LPFQ-filter of a bounded Q-algebra T. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ satisfy $\alpha_1 \leq \alpha_2$, $\beta_1 \geq \beta_2$, and $\alpha_i^2 + \beta_i^2 \leq 1$, i=1,2. Then $C_p(\alpha_2,\beta_2) \subseteq C_p(\alpha_1,\beta_1)$.

Proof. Let $\xi \in C_p(\alpha_2,\beta_2)$. By the definition of the (α_2,β_2)-level set, $\mu_p(\xi) \geq \alpha_2$ and $\nu_p(\xi) \leq \beta_2$. Since $\alpha_1 \leq \alpha_2$, we obtain $\mu_p(\xi) \geq \alpha_1$. Moreover, since $\beta_1 \geq \beta_2$, we have $\nu_p(\xi) \leq \beta_1$. Therefore, $\xi \in C_p(\alpha_1,\beta_1)$. Hence, $C_p(\alpha_2,\beta_2) \subseteq C_p(\alpha_1,\beta_1)$. □

Proposition 3.4. Let P be a Pythagorean fuzzy set on T. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ satisfy

$\alpha_i^2 + \beta_i^2 \leq 1$, i=1,2. Then $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2) = C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\})$.

Proof. Let $\xi \in C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2)$. Then $\mu_p(\xi) \geq \alpha_1$, $\mu_p(\xi) \geq \alpha_2$, and $\nu_p(\xi) \leq \beta_1$, $\nu_p(\xi) \leq \beta_2$. Consequently, $\mu_p(\xi) \geq \max\{\alpha_1,\alpha_2\}$ and $\nu_p(\xi) \leq \min\{\beta_1,\beta_2\}$. Hence, $\xi \in C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\})$. Therefore, $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2) \subseteq C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\})$. Conversely, let $\xi \in C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\})$. Then $\mu_p(\xi) \geq \max\{\alpha_1,\alpha_2\}$, which implies $\mu_p(\xi) \geq \alpha_1$ and $\mu_p(\xi) \geq \alpha_2$. Similarly, $\nu_p(\xi) \leq \min\{\beta_1,\beta_2\}$, which implies $\nu_p(\xi) \leq \beta_1$ and $\nu_p(\xi) \leq \beta_2$. Thus $\xi \in C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2)$. Combining the two inclusions, we obtain the stated equality. □

Theorem 3.5. Let P be an LPFQ-filter of a bounded Q-algebra T. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ satisfy $\alpha_i^2 + \beta_i^2 \leq 1$, i=1,2. If $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2) \neq \emptyset$, then $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2)$ is a Q-filter of T.

Proof. By Proposition 3.4, we have $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2) = C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\})$. Since $\alpha_i^2 + \beta_i^2 \leq 1$ and $\alpha=\max\{\alpha_1,\alpha_2\}$, $\beta=\min\{\beta_1,\beta_2\}$, we have $\alpha^2+\beta^2\leq 1$. Moreover, by assumption, $C_p(\alpha,\beta)\neq\emptyset$. Since P is an LPFQ-filter, Definition 3.1 implies that every nonempty (α,β)-level set of P is a Q-filter of T. Therefore, $C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2)$ is a Q-filter of T. □

Theorem 3.6. Let $(T,*,0)$ be a bounded Q-algebra with unit e, and let $P=(\mu_p,\nu_p)$ be a Pythagorean fuzzy set on T. For $\xi,\zeta \in T\backslash\{0\}$, put $\Phi(\xi,\zeta) = (\xi^c * \zeta)^c$. Then P is an LPFQ-filter of T if and only if $\mu_p(\xi) \geq \min\{\mu_p(\zeta), \mu_p(\Phi(\xi,\zeta))\}$ and $\nu_p(\xi) \leq \max\{\nu_p(\zeta), \nu_p(\Phi(\xi,\zeta))\}$ for all $\xi,\zeta \in T\backslash\{0\}$.

Proof. Suppose first that P is an LPFQ-filter of T. Let $\xi,\zeta \in T\backslash\{0\}$, and put

$$\alpha = \min\{\mu_p(\zeta), \mu_p(\Phi(\xi,\zeta))\}, \quad \beta = \max\{\nu_p(\zeta), \nu_p(\Phi(\xi,\zeta))\}.$$

Since P is a Pythagorean fuzzy set, $\mu_p(\zeta)^2 + \nu_p(\zeta)^2 \leq 1$ and $\mu_p(\Phi(\xi,\zeta))^2 + \nu_p(\Phi(\xi,\zeta))^2 \leq 1$. It follows that $\alpha^2 + \beta^2 \leq 1$. Indeed, if $\mu_p(\zeta) \leq \mu_p(\Phi(\xi,\zeta))$, then $\alpha=\mu_p(\zeta)$. If also $\nu_p(\zeta) \geq \nu_p(\Phi(\xi,\zeta))$, then $\alpha^2+\beta^2=\mu_p(\zeta)^2+\nu_p(\zeta)^2\leq 1$. In the remaining cases, the same argument follows by interchanging the two elements. Hence $C_p(\alpha,\beta)$ is an admissible level set.

By the definition of α and β, $\mu_p(\zeta) \geq \alpha$, $\nu_p(\zeta) \leq \beta$, and $\mu_p(\Phi(\xi,\zeta)) \geq \alpha$, $\nu_p(\Phi(\xi,\zeta)) \leq \beta$. Therefore, $\zeta, \Phi(\xi,\zeta) \in C_p(\alpha,\beta)$. Since P is an LPFQ-filter, every nonempty admissible level set is a Q-filter. Hence $\xi \in C_p(\alpha,\beta)$. Consequently, $\mu_p(\xi) \geq \alpha$ and $\nu_p(\xi) \leq \beta$. Thus, $\mu_p(\xi) \geq \min\{\mu_p(\zeta), \mu_p(\Phi(\xi,\zeta))\}$. And $\nu_p(\xi) \leq \max\{\nu_p(\zeta), \nu_p(\Phi(\xi,\zeta))\}$.

Conversely, suppose that $\mu_p(\xi) \geq \min\{\mu_p(\zeta), \mu_p(\Phi(\xi,\zeta))\}$ and $\nu_p(\xi) \leq \max\{\nu_p(\zeta), \nu_p(\Phi(\xi,\zeta))\}$ for all $\xi,\zeta \in T\backslash\{0\}$. Let $C_p(\alpha,\beta) \neq \emptyset$ be an arbitrary admissible level set. Suppose that $\zeta, \Phi(\xi,\zeta) \in C_p(\alpha,\beta)$. Then $\mu_p(\zeta) \geq \alpha$ and $\mu_p(\Phi(\xi,\zeta)) \geq \alpha$. Therefore, $\min\{\mu_p(\zeta), \mu_p(\Phi(\xi,\zeta))\} \geq \alpha$ and hence $\mu_p(\xi) \geq \alpha$. Similarly, $\nu_p(\zeta) \leq \beta$ and $\nu_p(\Phi(\xi,\zeta)) \leq \beta$. Thus $\max\{\nu_p(\zeta),\nu_p(\Phi(\xi,\zeta))\} \leq \beta$, and consequently $\nu_p(\xi) \leq \beta$. Therefore, $\xi \in C_p(\alpha,\beta)$. Hence every nonempty admissible (α,β)-level set of P is a Q-filter of T. By Definition 3.1, P is an LPFQ-filter of T. Therefore, the desired equivalence holds. □

Remark 3.7. Theorem 3.6 provides a direct algebraic characterization of LPFQ-filters. In particular, it allows the LPFQ-filter property to be verified through the membership and non-membership functions without checking the corresponding level sets individually.

4 . Conclusions

In this work, introduced notion of level Pythagorean fuzzy Q-filter (LPFQ-filter) in a bounded Q-algebra. The proposed notion is defined through the nonempty (α,β)-level sets of a Pythagorean fuzzy set, requiring each such level set to be a Q-filter.

A non-constant example was presented to demonstrate that the proposed notion is nontrivial. We then established fundamental properties of the corresponding level sets. In particular, we proved their nestedness with respect to the membership and non-membership thresholds and characterized the intersection of two level sets by

$$C_p(\alpha_1,\beta_1) \cap C_p(\alpha_2,\beta_2) = C_p(\max\{\alpha_1,\alpha_2\}, \min\{\beta_1,\beta_2\}).$$

As a consequence, whenever this intersection is nonempty, it is again a Q-filter of the underlying bounded Q-algebra.

These results establish a basic connection between Pythagorean fuzzy level structures and Q-filters and provide a foundation for further studies of generalized fuzzy filter structures in Q-algebras. Future work may investigate additional algebraic operations, generalized fuzzy filter notions, and corresponding ideal structures.

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