A New Multi Objective Model for Single Machine Scheduling Problems

Authors Names

Tahani Jabbar Khraibet

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ABSTRACT

This paper presents an exact method for the single machine scheduling problem with the objective of minimizing the total weight completion time and the sum of maximum earliness and maximum tardiness $\left(1 // \sum_{j=1}^{k} w_j C_j + ET_{max}\right)$. To obtain the optimal solution, a Branch and Bound (BAB) algorithm is developed using one effective upper bound (UB) and a lower bound (LB) to reduce the search space and improve computational efficiency. The computational results show that the suggested approach is able to generate optimal solutions for problem instances with up to 24 jobs in a tolerable computational time. The Matlab program was used to implement all algorithm.

1. Introduction

Scheduling is a calendar of when to do things, when to use resources or when to use facilities. Scheduling process can be considered as the execution step of production planning, and as a continual activity in the life of production systems . It constitutes a mathematical formula based on data, which is created by combining resources (machines), tasks (jobs) and standards of objectivity within specified constraints [1]. Scheduling theory refers to the machine scheduling problem (MSP), which is applicable to a variety of industries, including manufacturing facilities [2]. A combinatorial optimization issue known as Machine Scheduling Problems (MSPs) is employed in a variety of industries, such as manufacturing and services, to facilitate decision-making. No solution can be considered preferable to another due to the fact that there are numerous optimal solutions for each objective function. Non-dominated solutions are the optimal solutions in this context [3].

Each job j, $j \in N$, where $N = \{1,2,\dots,n\}$ requires a processing time $p_j$and should ideally be completed on it is due date d_j. And a positive weighted w_j on the machine. Our objective is to find a schedule that minimize the total weight completion time and the sum of maximum earliness and maximum tardiness. of job j can be respectively defined as: $\sum_{j=1}^{k} w_j C_j$ where $C_j = \sum_{j=1}^{k} p_j$ and $ET_{max} = E_{max}\, and\, T_{max}$ are given by: $E_j = max\{d_j - c_j, 0\}$ & $T_j = max\{c_j - d_j, 0\}$. The total weight completion time are minimized SWPT (the shortest weighted processing time) rule is optimum to [4. The sum of maximum earliness and maximum tardiness are referred to as EDD and MST rules respectively [5]. The classification of many scheduling problems as NP-hard necessitates the use of single machine to obtain optimal solution [6-7]. To present several studies from the literature including machine scheduling problems with one or more of the performance criteria defined in our challenge. Amin-Nayeri and Moslehi [8] suggested $\mathbf{1} // \mathbf{E}\mathrm{T}_{\mathbf{max}}$ problem then used the BAB method. Mahnam and Ghomi [9] considered the BAB method to solve the problem $1/r_j/E_{max} + T_{max}$. Györgyi and Kis [10] suggested to solve the problem $\mathbf{1} // \sum_{j\in J} w_j C_j$. Amin and Ramadan [11] used a heuristic method to solve the problem $\mathbf{1} // \mathbf{E}\mathrm{T}_{\mathbf{max}}$ to find near optimal Solution. Chen et.al [12] used a heuristic method to solve the problem $\mathbf{1} // \sum_{j\in J} w_j C_j$. Neamah et al. used a BAB method to solve two problems $1 //\sum_{j=1}^{n}(\mathrm{C_j}, \mathrm{E_j}, T_{max})$ and $1 //\sum_{j=1}^{n}(\mathrm{C_j} + \mathrm{E_j}) + T_{max})$ [13]. The solutions for SMSP problems have become more and more complex in recent decades. Application to scheduling problems on a single machine with multi- objectives has huge barriers because of the difficulty in handling several objectives [14]. Consequently, Branch and Bound were introduced to achieve optimal solutions for the problem at hand [15].

Department of Mathematics Thi-Qar Directorate of Education, Ministry of Education, Thi-Qar, Iraq,., E-Mail: tahani@utq.edu.iq

This study produced various contributions, which can be summarized as follows:

- The study offered new mathematical model for multi-objective function $\left(1 // \sum_{j=1}^{k} w_j C_j + ET_{max}\right)$ based on a single machine scheduling problem.
- A new BAB method was proposed to solve the multi-objective function with suggested upper and lower bounds

The remaining parts of this work are organized as follows: In Section 2 we describe the mathematical model of the two problems. Section 3, gives the exact method (BAB). Section 4 presents the results and comments. Finally, in section 5 the paper is concluded and some ideas for future work are offered.

2. Mathematical model

A collection of multi-objective problems is presented in this section, each containing n tasks that are intended to solve the single machine scheduling problem.

2.1 Notations

The mathematical formulation of the notations is given in Table 1.

Symbol	Descriptions
N	$\{1,2,3,\dots,k\}$
π	is the set of all schedules
p_j	Processing time
w_j	Weight
d_j	Due date
C_j	Completion time
BAB	Branch-and Bounded
MSP	Machine Scheduling Problem
LB	Lower bound
UB	Upper bound
NP	Non-deterministic Polynomial-time hard.
MST	minimum slack times
EDD	Early due date
OPT	Optimal value as determined by BAB.
Status	$= \begin{cases} 0 & \text{if the problem is solve} \\ 1 & \text{if the problem isn't solved} \end{cases}$
Nodes	The number generated
Time	Time of computation (second)
*	The optimal value is given by the upper bound.
**	The optimal value is given by the lower bound

2.2. Objective functions

The multi-objective function was presented in this subsection as follows: Two objective functions were introduced in this study to reduce that minimize the total weight completion time and the sum of maximum earliness and maximum tardiness $\left(1 // \sum_{j=1}^{k} w_j C_j + ET_{max}\right)$ it referred to the (F_T) problem, which can be expressed as follows:

$$\boldsymbol{F_T = min_{\pi \in S}\{M_{(\pi)}\} = min_{\pi \in S}\left\{\sum_{j=1}^{k} w_j C_j + ET_{max}\right\}} \qquad (\mathbf{1})$$

s.t

$$\left.\begin{array}{ll} C_{\pi(j)} \geq p_{\pi(j)} & j = 1,2,3,\dots,k \\ C_{\pi(j)} \geq C_{\pi(j-1)} + p_{\pi(j)} & j = 2,3,\dots,k \\ E_{\pi(j)} \geq d_{\pi(j)} - C_{\pi(j)} & j = 1,2,3,\dots,k \\ T_{\pi(j)} \geq C_{\pi(j)} - d_{\pi(j)} & j = 1,2,3,\dots,k \\ d_{\pi(j)} > 0\ , w_{\pi_{(j)}} \geq 0, p_{\pi_{(j)}} > 0, E_{\pi_{(j)}} \geq 0, T_{\pi_{(j)}} \geq 0 & j = 1,2,3,\dots k \end{array}\right\} \qquad (2)$$

3. Exact methods

3.1 Branch and Bound method

Our BAB method is based on the rule of forward sequencing branching, where the nodes in the search tree represent the first partial sequence with jobs in the initial locations [16]. The cost of unsequenced jobs is the derived lower bound ,whereas the objective function calculates the cost of scheduling jobs. The LB at each node is the sum of these two costs. A dominating node may exist at any level of the BAB. The sequence is examined to determine whether the branching produces a complete sequence of tasks. If its value is less than the current (UB), the current (UB) is reset to that value. A method known as BAB has been developed by a few researchers [17] [18].

3.1.1 Upper Bound (UB)

The three upper bounds are introduced in this subsection as follows:

1. UB_1: Depends on $\mu =$ WSPT rule, that is ordering the jobs in non-increasing order of $\frac{w_{(\mu)1}}{p_{(\mu)1}} \leq \frac{w_{(\mu)2}}{p_{(\mu)2}} \leq \cdots \leq \frac{w_{(\mu)k}}{p_{(\mu)k}}$ then:

$$UB_1 = \sum_{j=1}^{k} w_{\mu(j)} C_j + ET_{max}(\mu) \qquad (3)$$

2. UB_2: Depends on $\rho = SPT$ rule, that is sequencing the jobs in non-decreasing order of processing time $p_1 \leq p_2 \leq \cdots \leq p_k$.then:

$$UB_2 = \sum_{j=1}^{k} w_{\rho(j)} C_j + ET_{max}(\rho) \qquad (4)$$

3. UB_3: Depends on $\sigma = EDD$ rule, that is sequencing the jobs in non-decreasing order of due dates $d_1 \leq d_2 \leq \cdots \leq d_k$. then:

$$UB_3 = \sum_{j=1}^{k} w_{\sigma(j)} C_j + ET_{max}(\sigma) \qquad (5)$$

4. UB_3: Depends on $\pi = MST$ rule, that is sequencing the jobs in non-decreasing order of stare time $S_1 \leq S_2 \leq \cdots \leq S_k$. then:

$$\mathrm{UB}_4 = \sum_{j=1}^{k} w_{\pi(j)} C_j + ET_{max}(\pi) \qquad (6)$$

$$\mathrm{UB} = \min\{\mathrm{UB}_1, \mathrm{UB}_2, \mathrm{UB}_3, \mathrm{UB}_4\} \qquad (7)$$

3.1.2 Lower Bound (LB)

The lower bound (LB) is based on decomposing problem (F_T) into two subproblems (V_1) and (V_2) which can be stated as follows:

$$LB_1 = V_1 = min_{s\in S}\left\{\sum_{j=1}^{k}(w_j C_j)\right\}$$

s.t

$$\left.\begin{array}{ll} C_{\pi(j)} \ge p_{\pi(j)} & j = 1,2,3,\dots,k \\ C_{\pi(j)} \ge C_{\pi(j-1)} + p_{\pi(j)} & j = 2,3,\dots,k \\ w_{\pi_{(j)}} \ge 0, p_{\pi_{(j)}} > 0 & j = 1,2,3,\dots k \end{array}\right\} \qquad (8)$$

$$LB_2 = V_2 = min_{s\in S}\{ET_{max}\} = E_{max} + T_{max}$$

s.t

$$\left.\begin{array}{ll} C_{\pi(j)} \ge p_{\pi(j)} & j = 1,2,3,\dots,k \\ C_{\pi(j)} \ge C_{\pi(j-1)} + p_{\pi(j)} & j = 2,3,\dots,k \\ E_{\pi(j)} \ge d_{\pi(j)} - C_{\pi(j)} & j = 1,2,3,\dots,k \\ T_{\pi(j)} \ge C_{\pi(j)} - d_{\pi(j)} & j = 1,2,3,\dots,k \\ E_{\pi_{(j)}} \ge 0, T_{\pi_{(j)}} \ge 0, d_{\pi(j)} > 0 & j = 1,2,3,\dots k \end{array}\right\} \qquad (9)$$

Then: $LB = LB_1 + LB_2$ (10)

The following steps of Algorithm

Step1. Input: $n, p_h, d_h \; and \; w_h$ for j=1,2,3,…,k.
Step2. Calculate the UB at the parent node of the search tree: utilizing relations (3),(4),(5) and (6). Then the UB can be computed as in relation (7).
Step3. Determine LB for each node of the BAB search tree by combining the estimated contribution of the unscheduled jobs with the contribution of the scheduled jobs, arranged according to the WSPT and EDD rules in relations (8),(9) and (10).
Step4. The nodes are branched with $LB \le UB$ for level j.
Step5. If $j < k$ go to step 4.
Step6. The optimal solution is determined when $j = k$ at the final level.
Step7. STOP.

4. Computational results

To evaluate the proposed methods, we performed extensive computational experiments on a collection of common single machine scheduling problems ranging in size from 3 to 24 jobs. The variety of issue sizes allows for broad testing of the Branch and Bounded (BAB) algorithms in terms of scalability and computing performance. For $n \le 24$, the BAB gives minimum values for the problem was not solved when $n > 24$ in Table (2). The stopping condition for the BAB algorithm has been found, and we consider the problem to be unresolved (state 1) because the BAB method stops after a specified period of time, which in this case is 1800 seconds (i.e. 30 minutes) all utilized the same instances for a fair comparison. A workstation with an Intel Core i7 (3.4 GHz) processor and 16

GB RAM was used to execute all algorithms, which were programd in MATLAB R2024a.

Table (2): **:** Using BAB for the ($\boldsymbol{F_T}$) problem to find the optimal solution

N	Ex	OPT	UB	LB	Node	Time	Status
3	1	24.631	24.631*	24.631**	0	0	0
	2	22.6907	22.6907*	21.0033	14	0.00171	0
	3	17.7636	17.7636*	17.3052	14	0.00265	0
	4	29.461	29.461*	29.461**	0	0.0237	0
	5	23.347	23.347*	21.7292	14	0.00179	0
6	1	56.988	56.988*	55.9386	452	0.02779	0
	2	87.6323	87.6323*	87.6212	81	0.00462	0
	3	66.6356	66.6356*	62.0492	130	0.00792	0
	4	52.4686	53.094	45.7251	256	0.02838	0
	5	44.0641	44.0641*	38.2808	170	0.01661	0
9	1	116.926	116.926*	115.894	548	0.05523	0
	2	59.9902	59.9902*	59.9273	541	0.04393	0
	3	89.2231	89.2231*	81.0298	472	0.03111	0
	4	89.2735	93.7428	87.5717	13039	1.91063	0
	5	142.845	142.845*	142.845**	0	0	0
12	1	116.682	124.994	108.469	13612	0.80882	0
	2	177.723	177.723*	177.7229	930	0.04269	0
	3	173.665	173.665*	169.665	17243	0.96004	0
	4	138.75	138.766	138.741	964	0.04007	0
	5	130.715	143.051	128.982	17972	0.96066	0
15	1	156.497	156.498	156.49	2287	0.32055	0
	2	170.719	171.232	170.521	4542	0.49602	0
	3	139.252	154.12	136.015	20173	1.21721	0
	4	176.848	180.29	176.26	52900	3.28133	0
	5	161.449	176.518	158.036	32821	1.98433	0
17	1	153.538	157.803	147.922	323939	37.0033	0
	2	230.568	230.528	230.565	4173	0.26704	0
	3	291.559	291.559*	291.559**	0	0	0
	4	220.437	222.437*	220.435	3331	0.19594	0
	5	168.924	170.908	166.863	5704781	390.02	0
21	1	181.546	204.651	175.171	396253	30.3092	0
	2	226.842	246.795	226.794	518350	39.7521	0
	3	276.84	280.84*	276.788	11138	0.824	0
	4	186.95	201.922	175.42	269993	20.6146	0
	5	238.02	253.932	253.932	23489641	1800	1
23	1	237.901	245.882	237.866	45524	6.30542	0
	2	193.086	210.184	189.654	3311682	276.266	0
	3	292.109	299.114	292.108	360485	30.614	0
	4	170.426	170.541	166.125	41011	3.36209	0
	5	197.392	212.953	182.764	20925543	1800.1	1
	1	342.626	344.552	342.552	2375518	241.14	0
	2	338.708	344.708	338.708	11091	0.93933	0
	3	266.303	283.001	260.576	19356487	1800.1	1

	4	251.37	278.97	250.3	148678	13.9024	0
	5	262.27	289.984	255.211	19296254	1800.1	1
24	1	320.136	345.583	320.021	17719828	1800	1
	2	298.433	324.387	298.377	1525686	153.765	0
	3	285.128	321.918	278.514	17808736	1800.1	1
	4	359.522	367.898	357.898	18167237	1800	1
	5	407.199	411.19	407.19	4750232	491.484	0

5. Conclusions and future work

In this paper, a novel mathematical model for multi-objective function in single machine scheduling is proposed $\left(1 // \sum_{j=1}^{k} w_j C_j + ET_{max}\right)$, this problem is considered to be NP-hard. We used BAB algorithm to find the optimal solution up to n=24 for comparing the results in terms of accuracy and computing time. Additionally, several recommendations for future research were presented, based on a multi-criteria approach for a machine scheduling problem: In future work we will propose certain challenges for discussion and analysis:

1. $F_2 // \sum_{j=1}^{k} w_j C_j, T_{max}$
2. $1 // \sum_{j=1}^{k} w_j C_j, ET_{max}, \sum_{j=1}^{k} V_j$

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