

T_2 – Splitting Points in Topological Spaces

Authors Names	ABSTRACT
Abedal-Hamza M. Hamza ^a , Z. D. Al-Nafie ^b , Yiezi K. Al Talkany ^a Publication date: 25 / 8/2026 Keywords: <i>topological space,</i> <i>T_2- space, closure of a set,</i> <i>ideal.</i>	In this paper, we introduced a new splitting for points in topological spaces called it an lpT_2 – space using the concept of the derived set $d(\mathcal{A})$ of a set $\mathcal{A}$. We proved that this property is both a hereditary property and a topological property. The continuous image of an lpT_2 –space needs not be an lpT_2 – space. Also, we proved that the product space $(\mathbb{X} \times \mathbb{Y}, \tau_\rho)$ of two lpT_2 – spaces is lpT_2 –space if and only if each of $(\mathbb{X}, \mathbb{T})$ and $(\mathbb{Y}, \mathcal{S})$ is lpT_2 –space. We proved that every classical lpT_2 – space is a completely ideal Hausdorff space.

1. Introduction

General topology has its roots in real and complex analysis, which made important uses of the interrelated concepts of open set, of closed set, and of a limit point of a set. L. E. J. Brouwer, Georg Cantor, Maurice Fréchet and Felix Hausdorff introduced an axiomatic foundation for topological spaces and thereby initiated the general study of topology as an abstract mathematics discipline [3].

In 1872, Cantor used his new term “limit point” to define the “derived set” of a set $\mathcal{A}$, i.e., the set of all limit points of $\mathcal{A}$. In 1884, Cantor first defined the notion of closed set as a set that contains all its limit points. Jordan followed Cantor in defining the derived set of $\mathcal{A}$ to be the set of all limit points of $\mathcal{A}$. In 1914, Hausdorff defined ω to be a limit point of a set $\mathcal{A}$ if every neighborhood of ω contained

infinitely many points of $\mathcal{A}$. In 1922, Kuratowski gave the closure of $\mathcal{A}$ as $\mathcal{A} \cup d(\mathcal{A})$ [4].

Earlier separation axioms had been discovered and were called *Trennungsaxiome* (German) by P. Alexandroff and H. Hopf. Hence, these axioms are known as $T_n, n = 0, 1, 2, \dots, 5$. Often these axioms have alternate names such as Hausdorff, normal, Tychonoff, and so on [9].

In 1914, Hausdorff defined the concept of Hausdorff space (T_2 space). In 1935, Aleksandrov and Hopf discussed in great detail all of the separation axioms T_2, T_3 , and T_4 , as well as the weaker separation axioms T_1 and T_0 [4]. In 1975, S. N. Maheshwari and R. Prasad used semi-open sets to generalize T_1, T_0 and T_2 spaces to semi- T_0 , semi- T_1 and semi- T_2 [7]. In 1990, A. Kar, and P. Bhattacharyya used semi-open sets to generalize T_1, T_0 and T_2 spaces to pre- T_0 , pre- T_1 and pre- T_2 [1]. In 2018, A. M. Hamza and Z. N. Hameed used the boundary of sets to introduce new types of T_2 spaces called ZT2 space and semi-ZT2 space [2]. Throughout this paper $cl(A)$ and $d(A)$ will denote

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closure of the set A and the derived set of the set A respectively also, a space means a topological space.

2. Preliminaries

Definition 2.1. [3] A space $(\mathbb{X}, \mathbb{T})$ is Hausdorff (T_2) – space iff for every two distinct points in $\mathbb{X}$, there exist disjoint open sets, each containing one of the points.

Definition 2.2. [3] A property of topological spaces that is preserved by homeomorphism is said to be a topological property.

Definition 2.3. [7] A space $(\mathbb{X}, \mathbb{T})$ is called pre- T_2 iff for every two distinct points in $\mathbb{X}$, there exist disjoint preopen sets, each containing one of the points.

Definition 2.4. [1] A space $(\mathbb{X}, \mathbb{T})$ is called semi- T_2 iff for every two distinct points in $\mathbb{X}$, there exist disjoint semi-open sets, each containing one of the points.

Definition 2.5. [8] An ideal λ on a nonempty set $\mathbb{X}$ is a nonempty collection of the power set of $\mathbb{X}$ such that *i.* $A_1, A_2 \in \lambda \Rightarrow A_1 \cup A_2 \in \lambda$, *ii.* $A_1 \in \lambda \ \& \ A_2 \subseteq A_1 \Rightarrow A_2 \in \lambda$. If λ is an ideal of a space $(\mathbb{X}, \mathbb{T})$, then $(\mathbb{X}, \mathbb{T}, \lambda)$ is called ideal space or ideal topological space.

Definition 2.6. [6] An ideal topological space $(\mathbb{X}, \mathbb{T}, \lambda)$ is called completely ideal Hausdorff space if for each pair of distinct points $x_1, x_2 \in \mathbb{X}$, there are two open sets $\mathbb{U}, \mathbb{V}$ such that $x_1 \in \mathbb{U}, x_2 \in \mathbb{V}$, and $cl(\mathbb{U}) \cap cl(\mathbb{V}) \in \lambda$.

Theorem 2.7. [1] Let $(\mathbb{X}, \mathcal{T})$ and $(\mathbb{Y}, \mathcal{S})$ be spaces. If $\mathbf{B}$ is a basis for $\mathcal{T}$ and $\mathbf{C}$ is a basis for $\mathcal{S}$, then $\Omega = \{B \times C \mid B \in \mathbf{B}, C \in \mathbf{C}\}$ is a basis for the product topology τ_p on $\mathbb{X} \times \mathbb{Y}$.

Theorem 2.8. [5] Let $(\mathbb{X}, \mathcal{T})$ and $(\mathbb{Y}, \mathcal{S})$ be spaces and $A \subseteq \mathbb{X}, B \subseteq \mathbb{Y}$. Then

$$d(A \times B) = (d(A) \times cl(B)) \cup (cl(A) \times d(B))$$

3. Main Results

Definition 3.1. A topological space $(\mathbb{X}, \mathbb{T})$ is called an lpT_2 – space iff for each $x_1, x_2 \in \mathbb{X}, x_1 \neq x_2$, there exist two disjoint open sets $\mathbb{U}, \mathbb{V}$ such that $x_1 \in \mathbb{U}, x_2 \in \mathbb{V}$, and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$.

Examples 3.2.

- i. The space of real numbers with the usual topology, $(\mathbb{R}, \tau_u)$, is an lpT_2 – space.
- ii. The space of real numbers with the cofinite topology, $(\mathbb{R}, \tau_c)$, is not lpT_2 –space since the derived set of any open set is $\mathbb{R}$.

Theorem 3.3. The property of being lpT_2 – space is a hereditary property.

Proof: Let $(\mathbb{X}, \mathbb{T})$ be an lpT_2 – space and let $(\mathbb{Y}, \mathbb{T}_Y)$ be a subspace of $(\mathbb{X}, \mathbb{T})$.

Let $y_1, y_2 \in \mathbb{Y}, y_1 \neq y_2$.

Since $(\mathbb{X}, \mathbb{T})$ is an lpT_2 – space, there exist two disjoint open sets $\mathbb{U}, \mathbb{V}$ such that $y_1 \in \mathbb{U}, y_2 \in \mathbb{V}$, and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$. The sets $\mathbb{G} = \mathbb{Y} \cap \mathbb{U}$ and $\mathbb{W} = \mathbb{Y} \cap \mathbb{V}$ are disjoint open sets in $\mathbb{Y}$, $y_1 \in \mathbb{G}, y_2 \in \mathbb{W}$.

$$\begin{aligned} \text{Now, } d(\mathbb{G}) \cap d(\mathbb{W}) &= d(\mathbb{Y} \cap \mathbb{U}) \cap d(\mathbb{Y} \cap \mathbb{V}) \\ &\subseteq (d(\mathbb{Y}) \cap d(\mathbb{U})) \cap (d(\mathbb{Y}) \cap d(\mathbb{V})) \\ &= d(\mathbb{Y}) \cap (d(\mathbb{U}) \cap d(\mathbb{V})) \\ &= d(\mathbb{Y}) \cap \emptyset \\ &= \emptyset \end{aligned}$$

Hence, $(\mathbb{Y}, \mathbb{T}_Y)$ is an lpT_2 – space.

Remark 3.4. The continuous image of an lpT_2 – space needs not be an lpT_2 – space:

The function $f: (\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, \tau_c), f(x) = x$ is continuous and $(\mathbb{R}, \tau_u)$ is an lpT_2 – space, but $(\mathbb{R}, \tau_c)$ is not lpT_2 –space

Theorem 3.5. The property of being lpT_2 – space is a topological property.

Proof: Let $(\mathbb{X}, \mathcal{T})$ be an lpT_2 – space and it is a homomorphic to a topological space $(\mathbb{Y}, \mathcal{S})$.

Then there exists a homeomorphism $f: (\mathbb{X}, \mathcal{T}) \rightarrow (\mathbb{Y}, \mathcal{S})$.

Let $a, b \in \mathbb{Y}, a \neq b$. Then $f^{-1}(a), f^{-1}(b) \in \mathbb{X}$, and $f^{-1}(a) \neq f^{-1}(b)$.

Since $(\mathbb{X}, \mathcal{T})$ is an lpT_2 – space, then there exist disjoint open sets $\mathbb{U}, \mathbb{V}$ such that $f^{-1}(a) \in \mathbb{U}, f^{-1}(b) \in \mathbb{V}$ and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$.

Then $a \in f(\mathbb{U}), b \in f(\mathbb{V})$. Since f is an open function, then $f(\mathbb{U})$ and $f(\mathbb{V})$ are open sets in $\mathbb{Y}$.

Now, $f(\mathbb{U}) \cap f(\mathbb{V}) = f(\mathbb{U} \cap \mathbb{V}) = f(\emptyset) = \emptyset$.

$$d(f(\mathbb{U})) \cap d(f(\mathbb{V})) = f(d(\mathbb{U})) \cap f(d(\mathbb{V})) = f(d(\mathbb{U}) \cap d(\mathbb{V})) = f(\emptyset) = \emptyset.$$

Hence $(\mathbb{Y}, \mathcal{S})$ is an lpT_2 – space.

Theorem 3.7. $(\mathbb{X}, \mathcal{T})$ and $(\mathbb{Y}, \mathcal{S})$ are lpT_2 – spaces iff the product space $(\mathbb{X} \times \mathbb{Y}, \tau_\rho)$ is an lpT_2 – space.

Proof: Suppose that each of $(\mathbb{X}, \mathcal{T})$ and $(\mathbb{Y}, \mathcal{S})$ is an lpT_2 – space and let $(x_1, y_1), (x_2, y_2) \in \mathbb{X} \times \mathbb{Y}$, $(x_1, y_1) \neq (x_2, y_2)$. Suppose that $x_1 \neq x_2$.

Since $(\mathbb{X}, \mathcal{T})$ is an lpT_2 – space, then there exist disjoint open sets $\mathbb{U}, \mathbb{V}$ in $\mathbb{X}$ such that $x_1 \in \mathbb{U}, x_2 \in \mathbb{V}$ and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$.

Then $\mathbb{G} = \mathbb{U} \times \mathbb{Y}, \mathbb{W} = \mathbb{V} \times \mathbb{Y}$ are open sets in $\mathbb{X} \times \mathbb{Y}$, $(x_1, y_1) \in \mathbb{G}, (x_2, y_2) \in \mathbb{W}$,

$$\begin{aligned} \mathbb{G} \cap \mathbb{W} &= (\mathbb{U} \times \mathbb{Y}) \cap (\mathbb{V} \times \mathbb{Y}) \\ &= (\mathbb{U} \cap \mathbb{V}) \times (\mathbb{Y} \cap \mathbb{Y}) \\ &= \emptyset \times \mathbb{Y} \\ &= \emptyset \end{aligned}$$

and

$$d(\mathbb{G}) \cap d(\mathbb{W}) = d(\mathbb{U} \times \mathbb{Y}) \cap d(\mathbb{V} \times \mathbb{Y})$$

Using [Theorem 2.8], we have

$$\begin{aligned} d(\mathbb{G}) \cap d(\mathbb{W}) &= [(d(\mathbb{U}) \times cl(\mathbb{Y})) \cup (cl(\mathbb{U}) \times d(\mathbb{Y}))] \cap [(d(\mathbb{V}) \times cl(\mathbb{Y})) \cup (cl(\mathbb{V}) \times d(\mathbb{Y}))] \\ &= [(d(\mathbb{U}) \times \mathbb{Y}) \cup (cl(\mathbb{U}) \times d(\mathbb{Y}))] \cap [(d(\mathbb{V}) \times \mathbb{Y}) \cup (cl(\mathbb{V}) \times d(\mathbb{Y}))] \\ &\subseteq [(d(\mathbb{U}) \cup cl(\mathbb{U})) \times (\mathbb{Y} \cup d(\mathbb{Y}))] \cap [(d(\mathbb{V}) \cup cl(\mathbb{V})) \times (\mathbb{Y} \cup d(\mathbb{Y}))] \\ &= (cl(\mathbb{U}) \times cl(\mathbb{Y})) \cap (cl(\mathbb{V}) \times cl(\mathbb{Y})) \\ &= (cl(\mathbb{U}) \times \mathbb{Y}) \cap (cl(\mathbb{V}) \times \mathbb{Y}) \\ &= (cl(\mathbb{U}) \cap cl(\mathbb{V})) \times \mathbb{Y} \end{aligned}$$

Using $\mathbb{U} \cap \mathbb{V} = \emptyset$ and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$, we shall obtain $(cl(\mathbb{U}) \cap cl(\mathbb{V})) \times \mathbb{Y} = \emptyset \times \mathbb{Y} = \emptyset$.

So, $d(\mathbb{G}) \cap d(\mathbb{W}) = \emptyset$.

Hence, $(\mathbb{X} \times \mathbb{Y}, \tau_\rho)$ is an lpT_2 - space.

Now, suppose that $(\mathbb{X} \times \mathbb{Y}, \tau_\rho)$ is an lpT_2 - space. To prove that each of $(\mathbb{X}, \mathcal{T})$ and $(\mathbb{Y}, \mathcal{S})$ is an lpT_2 - space, let $x_1, x_2 \in \mathbb{X}, x_1 \neq x_2$, and $y \in \mathbb{Y}$. Then $(x_1, y), (x_2, y) \in \mathbb{X} \times \mathbb{Y}, (x_1, y) \neq (x_2, y)$. Since $(\mathbb{X} \times \mathbb{Y}, \tau_\rho)$ is an lpT_2 - space, then we have two disjoint open sets $\mathbb{U}, \mathbb{V}$ in $\mathbb{X} \times \mathbb{Y}$ such that $(x_1, y) \in \mathbb{U}, (x_2, y) \in \mathbb{V}$ and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$.

$\mathbb{U}$ and $\mathbb{V}$ are the union of basic open sets (basis elements) of the form $\mathbb{U} \times \mathbb{V}$ of $\mathbb{X} \times \mathbb{Y}$:

$$\mathbb{U} = \bigcup_{\alpha \in I} (\mathbb{W}_{x_\alpha} \times \mathbb{W}_{y_\alpha}), \mathbb{V} = \bigcup_{\beta \in J} (\mathbb{W}_{x_\beta} \times \mathbb{W}_{y_\beta})$$

So, (x_1, y) and (x_2, y) , belong to at least one of the basic open sets (basis elements), say, $\mathbb{W}_{x_1} \times \mathbb{W}_y$ and $\mathbb{W}_{x_2} \times \mathbb{W}_y$ respectively.

$x_1 \in \mathbb{W}_{x_1}, x_2 \in \mathbb{W}_{x_2}$ and since $\mathbb{W}_{x_1} \times \mathbb{W}_y \subseteq \mathbb{U}$ and $\mathbb{W}_{x_2} \times \mathbb{W}_y \subseteq \mathbb{V}$, and $\mathbb{U} \cap \mathbb{V} = \emptyset$, then

$$(\mathbb{W}_{x_1} \times \mathbb{W}_y) \cap (\mathbb{W}_{x_2} \times \mathbb{W}_y) = \emptyset.$$

$$(\mathbb{W}_{x_1} \times \mathbb{W}_y) \cap (\mathbb{W}_{x_2} \times \mathbb{W}_y) = (\mathbb{W}_{x_1} \cap \mathbb{W}_{x_2}) \times \mathbb{W}_y = \emptyset,$$

Since $\mathbb{W}_y \neq \emptyset$, then $\mathbb{W}_{x_1} \cap \mathbb{W}_{x_2} = \emptyset$.

Since $\mathbb{W}_{x_1} \times \mathbb{W}_y \subseteq \mathbb{U}$ and $\mathbb{W}_{x_2} \times \mathbb{W}_y \subseteq \mathbb{V}$, then $d(\mathbb{W}_{x_1} \times \mathbb{W}_y) \subseteq d(\mathbb{U})$ and $d(\mathbb{W}_{x_2} \times \mathbb{W}_y) \subseteq d(\mathbb{V})$.

Since $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$, then $d(\mathbb{W}_{x_1} \times \mathbb{W}_y) \cap d(\mathbb{W}_{x_2} \times \mathbb{W}_y) = \emptyset$.

Now,

$$d(\mathbb{W}_{x_1} \times \mathbb{W}_y) = (d(\mathbb{W}_{x_1}) \times cl(\mathbb{W}_y)) \cup (cl(\mathbb{W}_{x_1}) \times d(\mathbb{W}_y)) \text{ [Theorem 2.8]}$$

$$\text{So, } d(\mathbb{W}_{x_1}) \times cl(\mathbb{W}_y) \subseteq d(\mathbb{W}_{x_1} \times \mathbb{W}_y)$$

$$\text{Similarly, } d(\mathbb{W}_{x_2}) \times cl(\mathbb{W}_y) \subseteq d(\mathbb{W}_{x_2} \times \mathbb{W}_y).$$

This implies that

$$(d(\mathbb{W}_{x_1}) \times cl(\mathbb{W}_y)) \cap (d(\mathbb{W}_{x_2}) \times cl(\mathbb{W}_y)) \subseteq d(\mathbb{W}_{x_1} \times \mathbb{W}_y) \cap d(\mathbb{W}_{x_2} \times \mathbb{W}_y) = \emptyset.$$

But,

$$(d(\mathbb{W}_{x_1}) \times cl(\mathbb{W}_y)) \cap (d(\mathbb{W}_{x_2}) \times cl(\mathbb{W}_y)) = (d(\mathbb{W}_{x_1}) \cap d(\mathbb{W}_{x_2})) \times cl(\mathbb{W}_y) = \emptyset.$$

Since $cl(\mathbb{W}_y) \neq \emptyset$, then $d(\mathbb{W}_{x_1}) \cap d(\mathbb{W}_{x_2}) = \emptyset$.

Hence, $(\mathbb{X}, \mathcal{T})$ is an lpT_2 – space.

Similarly, we can prove that $(\mathbb{Y}, \mathcal{S})$ is an lpT_2 – space.

Theorem 3.7. If a classical space $(\mathbb{X}, \mathbb{T})$ is an clT_2 – space, then for any an ideal λ , the ideal topological space $(\mathbb{X}, \mathbb{T}, \lambda)$ is completely ideal Hausdorff space.

Proof: Let $(\mathbb{X}, \mathbb{T})$ be an lpT_2 – space and $(\mathbb{X}, \mathbb{T}, \lambda)$ is completely ideal Hausdorff space. Then for each $x, y \in \mathbb{X}, x \neq y$, there exist two disjoint open sets $\mathbb{U}, \mathbb{V}$ such that $x \in \mathbb{U}, y \in \mathbb{V}$, and $d(\mathbb{U}) \cap d(\mathbb{V}) = \emptyset$ which implies that $cl(\mathbb{U}) \cap cl(\mathbb{V}) = \emptyset$. Then $cl(\mathbb{U}) \cap cl(\mathbb{V}) \in \lambda$. Hence, $(\mathbb{X}, \mathbb{T}, \lambda)$ is completely ideal Hausdorff space.

4. Conclusions

lpT_2 – spaces is a new splitting for points in topological spaces depend on the derived of open sets.

These spaces have each of the hereditary property and the topological property. Also, the product space of two lpT_2 – spaces is lpT_2 –space if and only if each of the two spaces is an lpT_2 – space. We found that every classical lpT_2 – space is a completely ideal Hausdorff space.

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